ELEMENTA

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PRINCIPIA CALCULI DIFFERENTIALIS

DE FUNCTIONIBUS, DEQUE EARUM CONTINUITATE.

1. Si variabiles x, y, z, v, ... sunt inter se per certas relationes ita connexae, ut datis quibusdam inter illas v. gr. z, v, ..., inde possint caeterarum x, y, ... valores determinari, variabiles x, y, ... appellantur *functiones* variabilium z, v, ...; ipsae vero z, v, ... dicuntur *independentes*. Coordinatae v. gr. x, y lineae rectae, quae consideratur in plano, eam inter se habent relationem quae (172. I.° ex parte 2.^{a}) per aequationem $y=ax+b$ exprimitur: et quoniam datis valoribus x, proveniunt valores y determinati, et viceversa; iccirco duarum x, y altera poterit assumi ut alterius independentis functio. Si rectam lineam contemplamur in spatio, poterunt (182 ex p. 2.^{a}) relationes inter coordinatas x, y, z exhiberi per aequationes $y=az+b$, $x=a'z+b'$; et quia data z prodeunt valores determinari x, y, ideo poterunt x, y spectari ut functiones independentis z. Item coordinatae x, y, z superficiei planae sic inter se connectuntur, ut (180. I.° ex p. 2.^{a}) valeat aequatio $z=ax+a'y+b$; et quoniam datis binis, tertia prodit determinata, iccirco ex tribus x, y, z una quaevis erit caeterarum functio. Idipsum apparet in coordinatis linearum, et superficierum curvarum.

Generatim si m repraesentat numerum variabilium x, y, z, v, ..., et n numerum relationum, facile intelligitur fore $m-n$ numerum variabilium independentium, et n numerum functionum.

2. Si relationes inter variabiles exprimantur aequationibus minime resolutis quoad functiones pro incognitis habitas, hae vocantur *implicitae*. Quod si functionum valores dentur expressi immediate per variabiles independentes, vel tales obtineantur per aequationum resolutionem, functiones dicuntur explicitae. In aequatione v. gr. $y^2-2xy+m^2=0$ est y functio implicita quantitatis variabilis x; at facta resolutione evadet y functio explicita ipsius x, duplicemque habebit valorem, videlicet $y=x\pm\sqrt{x^2-m^2}$. Item (205 : 208 : 240. 6.° ex p. 1.^{a}) in aequatione $L(y)=x$ est y functio implicita quantitatis x; verum si fiat $L(y)=xL(e)=L(e^x)$, eruetur functio explicita $y=e^x$. Functiones explicitae unius vel plurium variabilium designari solent in hunc modum

$$F(x),\ f(x),\ \varphi(x),\ \chi(x),\ \text{et caet.}\ \ldots,$$
$$F(x,\ y,\ z,\ \ldots),\ f(x,\ y,\ z,\ \ldots),$$
$$\varphi(x,\ y,\ z,\ \ldots),\ \text{et caet.}\ \ldots$$

3. Functiones explicitae vocantur algebraicae (172. II° ex p. 2.^{a}) si variabiles independentes subjiciuntur dumtaxat primis Algebrae operationibus, videlicet additioni, subtractioni, multiplicationi, divisioni, et evectioni ad potentias fixas sive integras, sive fractas. Hinc functiones $a\sqrt{x}+bx+cx^2$, $a+\sqrt[3]{x^2}$, $\sqrt{a+mx+nx^2}$, $\dfrac{x\sqrt{a+cx^2}}{a+x}$ sunt omnes algebraicae: prima et secunda, in quibus variabilis x irrationalitate involvitur, dicuntur *irrationales*: tertia vero et quarta ex opposito *rationales*; eadem insuper tertia dicitur (156. ex p. 1.^{a}) integra, quia in ejus denominatore non reperitur variabilis x, quarta ex opposito fracta.

4. Functiones, quae non sunt algebraicae, vocan-

tur transcendentes : tales sunt quae continent variabiles vel signo logarithmico affectas, vel ut exponentes, vel ut lineas trigonometricas, aut respondentes arcus, quaeque peculiaribus etiam nominibus appellantur vel logarithmicae, vel exponentiales, vel trigonometricae.

5. Sit nunc functio

$$y=f(x).$$

Aucta, vel imminuta x, variabitur y; denotent Δy, Δx earum incrementa (vocantur *differentiae*, altera functionis y, altera quantitatis x): erit

$$y+\Delta y=f(x+\Delta x),$$

unde

$$\Delta y=f(x+\Delta x)-f(x),$$

in qua Δx assumi potest vel finita, vel infinitesima (250 ex p. 1.ª). Ponatur Δx infinitesima, et valor $f(x)$ permanere finitus ab $x=x_n$ ad $x=x_m$: functio y dicitur *continua* inter limites x_n, x_m quotiescumque singulis valoribus x inter limites illos interceptis respondet ejusmodi differentia

$$f(x+\Delta x)-f(x),$$

quae (131. 1.° ex p. 1.ª) inter ipsos x_n, x_m exsistat infinitesima.

Intelligatur describi curva aequationis

$$y=f(x).$$

Si $f(x)$ est continua inter certos limites, talis quoque erit curva, et vicissim: aucta vel imminuta abscissa x quantitate infinitesima Δx, variabitur inter eosdem limites ordinata y quantitate pariter infinitesima Δy ita, ut binis abscissis x, $x+\Delta x$ respondeant inter illos limites bina curvae puncta quorum distantia $\sqrt{\Delta x^2+\Delta y^2}$ erit et ipsa infinitesima, vergens nimirum indefinite una cum Δx, Δy ad limitem $=0$.

Ad haec: quando $f(x)$ dicetur esse continua in viciniis cujusdam peculiaris valoris qui tribuitur variabili x, id ita intelligendum erit ut ipsa $f(x)$ maneat continua inter binos limites peculiarem illum valorem comprehendentes, utcumque parum de caetero isti limites ab se invicem distent.

DE DIFFERENTIALIBUS FUNCTIONUM QUAE AB UNICA PENDENT VARIABILI.

6. Exhibita per β arbitraria quantitatum infinitesimarum basi (250. ex p. 1.ª), sumptaque Δx infinitesima, pone

$$\Delta x = i\beta \text{ , unde } lim. \frac{\Delta x}{\beta} = lim. \, i :$$

fac insuper ut functio

$$y = f(x)$$

exsistat continua (5) intra quaedam confinia variabili x assignata; etsi Δy intra confinia illa vergit ad $lim. = 0$, attamen ratio

$$\frac{\Delta y}{\beta}$$

poterit (192. 2.° ex p. 2.ª) ad alium limitem vergere sive $>$, sive < 0: limites rationum

$$\frac{\Delta x}{\beta}, \frac{\Delta y}{\beta}$$

appellamus *differentialia*, alterum variabilis independentis x, alterum functionis y, designamusque per dx, dy, ut scribi possit

$$dx = lim. \frac{\Delta x}{\beta} = lim. \, i ,$$

$$dy = lim. \frac{\Delta y}{\beta} = lim. \frac{f(x+i\beta) - f(x)}{\beta} :$$

hinc

$$\frac{dy}{dx} = \frac{lim.\frac{\Delta y}{\beta}}{lim.i} = lim.\frac{\Delta y}{i\beta} = lim.\frac{\Delta y}{\Delta x}.$$

In determinandis functionum differentialibus versatur *Calculus differentialis*.

Proponantur invenienda differentialia functionum

$$a \pm x\,,\ ax\,,\ \frac{a}{x}\,,\ x^a\,,\ a^x\,,\ \mathrm{L}(x)\,,\ sin x\,,\ cos x.$$

1.°

$$\frac{\Delta y}{\beta} = \frac{(a \pm x \pm i\beta) - (a \pm x)}{\beta} = \pm\frac{i\beta}{\beta} = \pm i\,;$$

$$dy = d(a \pm x) = lim.\frac{\Delta y}{\beta} = \pm\, lim.i = \pm\, dx.$$

2.°

$$\frac{\Delta y}{\beta} = \frac{a(x+i\beta) - ax}{\beta} = \frac{ai\beta}{\beta} = ai\,;$$

$$dy = dax = lim.\frac{\Delta y}{\beta} = lim.\,ai = adx.$$

3.°

$$\frac{\Delta y}{\beta} = \frac{\frac{a}{x+i\beta} - \frac{a}{x}}{\beta} = -\frac{ai\beta}{x\beta(x+i\beta)} = -\frac{ai}{x(x+i\beta)};$$

$$dy = d\frac{a}{x} = lim.\frac{\Delta y}{\beta} = -\frac{adx}{x^2}.$$

4.°

$$\frac{\Delta y}{\beta} = \frac{(x+i\beta)^a - x^a}{\beta} = \frac{x^a[(1+\frac{i\beta}{x})^a - 1]}{\beta},$$

seu (244 ex p. 1.ª)

$$\frac{\Delta y}{\beta}=x^{a-2}[ai+\frac{\beta}{x}\left(\frac{a(a-1)i^2}{2}+\right.$$

$$\left.\frac{a(a-1)(a-2)i^3\beta}{2.3.x}+\frac{a(a-1)(a-2)(a-3)i^4\beta^2}{2.3.4x^2}+\ldots\right)];$$

ideo (236 ex p. 1.ª)

$$dy=dx^a=lim.\frac{\Delta y}{\beta}=ax^{a-1}dx.$$

5.°

$$\frac{\Delta y}{\beta}=\frac{a^{x+i\beta}-a^x}{\beta}=a^x\frac{a^{i\beta}-1}{\beta},$$

seu (241 ex p. 1.ª)

$$\frac{\Delta y}{\beta}=a^x[iL(a)+\beta\left(\frac{i^2L^2(a)}{2}+\right.$$

$$\left.\frac{i^3\beta L^3(a)}{2.3}+\frac{i^4\beta^2L^4(a)}{2.3.4}+\ldots\right)];$$

propterea (236 ex p. 1.ª)

$$dy=da^x=lim.\frac{\Delta y}{\beta}=a^xL(a)\,dx.$$

6.° (207 ex p. 1.ª)

$$\frac{\Delta y}{\beta}=\frac{L(x+i\beta)-L(x)}{\beta}=\frac{L(1+\frac{i\beta}{x})}{\beta},$$

seu (242 ex p. 1.ª)

$$\frac{\Delta y}{\beta}=\frac{i}{x}-\beta\left(\frac{i^2}{2x^2}-\frac{i^3\beta}{3x^3}+\frac{i^4\beta^2}{4x^4}-\ldots\right);$$

proinde (236 ex p. 1.^{a})

$$dy = dL(x) = lim. \frac{\Delta y}{\beta} = \frac{dx}{x}.$$

7.° (127. 2.° ex p. 2.^{a})

$$\frac{\Delta y}{\beta} = \frac{sin(x+i\beta)-sin x}{\beta} = \frac{2cos(x+\frac{1}{2}i\beta)sin\frac{1}{2}i\beta}{\beta};$$

seu (163 ex p. 2.^{a})

$$\frac{\Delta y}{\beta} = cos(x+\frac{1}{2}i\beta)[i-\beta^2(\frac{i^3}{2^3.3} - \frac{i^5\beta^2}{2^5.3.4.5} + \ldots)];$$

quocirca (236 ex p. 1.^{a})

$$dy = dsin x = lim. \frac{\Delta y}{\beta} = cos x\, dx.$$

8.° (127. 1.° ex p. 2.^{a})

$$\frac{\Delta y}{\beta} = \frac{cos(x+i\beta)-cos x}{\beta} = -\frac{2sin(x+\frac{1}{2}i\beta)sin\frac{1}{2}i\beta}{\beta},$$

seu (163 ex p. 2.^{a})

$$\frac{\Delta y}{\beta} = -sin(x+\frac{1}{2}i\beta)[i-\beta^2(\frac{i^3}{2^3.3} - \frac{i^5\beta^2}{2^5.3.4.5} + \ldots)],$$

ideo

$$dy = d\,cos\,x = lim. \frac{\Delta y}{\beta} = -sin\,x\,dx.$$

Ad haec: in 1.° casu $\frac{dy}{dx} = \pm 1$; in 2.° $\frac{dy}{dx} = a$;

in 3.° $\frac{dy}{dx} = -\frac{a}{x^2}$; in 4.° $\frac{dy}{dx} = ax^{a-1}$;

in 5°. $\frac{dy}{dx}=a^x L(a)$; in 6.° $\frac{dy}{dx}=\frac{1}{x}$; in 7.° $\frac{dy}{dx}$ $=\cos x$; in 8.° $\frac{dy}{dx}=-\sin x$.

Quisque videt $\frac{dy}{dx}$ seu *lim.* $\frac{\Delta y}{\Delta x}$ fore generatim novam functionem variabilis x : si ea denotatur per $f'(x)$, erit

$$\frac{dy}{dx}=f'(x),\ \text{et}\ dy=f'(x)\,dx:$$

functio $f'(x)$ appellari solet *derivata ex primitiva* $f(x)$. Caeterum quemadmodum β sic *lim.* $\frac{\Delta x}{\beta}$ idest differentiale variabilis independentis x haberi debet pro constanti et arbitraria quantitate.

7. Detur *functio functionis*

$$z=F[f(x)]=F(y)\ldots(h)$$

Cum habeamus

$$\frac{\Delta z}{\beta}=\frac{\Delta z}{\Delta y}\cdot\frac{\Delta y}{\beta},$$

erit (6)

$$dz=lim.\frac{\Delta z}{\Delta y}\cdot lim.\frac{\Delta y}{\beta}=F'(y)dy=F'(y)f'(x)dx.$$

Sic v. gr. data $z=L(\sin x)$, erunt $y=f(x)=\sin x$, $F(y)=L(y)$ proinde (6. 6.° 7.°) $F'(y)=\frac{1}{y}=\frac{1}{\sin x}$, $f'(x)=\cos x$, ideoque $dz=dL(\sin x)=\frac{\cos x}{\sin x}dx=$ $\cot x\,dx$.

8. Pone

$$f(x)=\varphi(x)+\chi(x)+\ldots;$$

habebis

$$f(x+i\beta)=\varphi(x+i\beta)+\chi(x+i\beta)+\ldots,$$

unde

$$\frac{\Delta y}{\beta}=\frac{f(x+i\beta)-f(x)}{\beta}=\frac{\varphi(x+i\beta)-\varphi(x)}{\beta}+$$

$$\frac{\chi(x+i\beta)-\chi(x)}{\beta}+\ldots;$$

et facto ad limites gradu (6),

$$dy=df(x)=d\varphi(x)+d\chi(x)+\ldots,$$

$$\frac{dy}{dx}=f'(x)=\varphi'(x)+\chi'(x)+\ldots.$$

Differentiale nimirum quantitatis coalescentis e summa plurium functionum obtinetur differentiando seorsim functiones ipsas, et differentialia inde orta colligendo in summam: idipsum valet de functionibus derivatis. Hinc v. gr.

$$d(a+a'x+a''x^2+a'''x^3+\ldots)=$$

$$(a'+2a''x+3a'''x^2+\ldots)\,dx:$$

terminus constans a non invenitur in differentiali, idemque provenit differentiale sive differentietur $a+a'x+a''x^2+\ldots$ sive $a'x+a''x^2+\ldots$

9. Denotent s, t, u, ... functiones quantitatis variabilis x, sitque

$$z=stu\ldots,\ \text{et consequenter}\ z^2=s^2t^2u^2\ldots;$$

sive $s, t, u, \ldots$ exsistant >0, sive <0, erit (207 ex p. 1.ª)

$$\mathrm{L}(z^2)=\mathrm{L}(s^2)+\mathrm{L}(t^2)+\mathrm{L}(u^2)+\ldots;$$

ex cujus differentiatione (7 : 6, 6.°)

$$\frac{d(z^2)}{z^2}=\frac{d(s^2)}{s^2}+\frac{d(t^2)}{t^2}+\frac{d(u^2)}{u^2}+\ldots,$$

seu (6. 4.°)

$$\frac{dz}{z}=\frac{ds}{s}+\frac{dt}{t}+\frac{du}{u}+\ldots.$$

Est autem

$$\frac{dz}{z}=\frac{d(stu\ldots)}{stu\ldots};$$

igitur

$$d(stu\ldots)=stu\ldots\left(\frac{ds}{s}+\frac{dt}{t}+\frac{du}{u}+\ldots\right).$$

Hinc

$d(st)=tds+sdt$, $d(stu)=tuds+sudt+stdu$, et caet. ...

10. Sit nunc

$$z=\frac{st\ldots}{uv\ldots}, \text{ et consequenter } z^2=\frac{s^2t^2\ldots}{u^2v^2\ldots};$$

erit (207 ex p. 1.^{a})

$$L(z^2)=L(s^2)+L(t^2)+\ldots-L(u^2)-L(v^2)-\ldots;$$

ex cujus differentiatione (7 : 6. 6°. 4.°)

$$\frac{dz}{z}=\frac{ds}{s}+\frac{dt}{t}+\ldots-\frac{du}{u}-\frac{dv}{v}-\ldots$$

Propterea

$$d\left(\frac{st\ldots}{uv\ldots}\right)=\frac{st\ldots}{uv\ldots}\left(\frac{ds}{s}+\frac{dt}{t}+\ldots-\frac{du}{u}-\frac{dv}{v}-\ldots\right).$$

Hinc

$$d\frac{s}{u} = \frac{ds}{u} - \frac{sdu}{u^2} = \frac{uds - sdu}{u^2}, \text{ et caet.} \ldots$$

Sic v. gr. (6. 7°. 8.° 6° : 7)

$$d\,tang\,x = d\frac{\sin x}{\cos x} = \frac{\cos x\,d\sin x - \sin x\,d\cos x}{\cos^2 x} =$$

$$\frac{\cos^2 x + \sin^2 x}{\cos^2 x}dx = \frac{dx}{\cos^2 x},\ d\cot x = d\frac{\cos x}{\sin x} = -\frac{dx}{\sin^2 x},$$

$$d\sec x = d\frac{1}{\cos x} = \frac{\sin x\,dx}{\cos^2 x} = \frac{tang\,x\,dx}{\cos x},\ d\,cosec\,x =$$

$$d\frac{1}{\sin x} = -\frac{\cos x\,dx}{\sin^2 x} = -\frac{\cot x\,dx}{\sin x},\ dL(tang\,x) =$$

$$dL\left(\frac{\sin x}{\cos x}\right) = \frac{d\frac{\sin x}{\cos x}}{\frac{\sin x}{\cos x}} = \frac{\frac{dx}{\cos^2 x}}{\frac{\sin x}{\cos x}} = \frac{dx}{\sin x\cos x},$$

et caet. . . . .

11. Habemus (6. 7.° 8° : 10)

$$dx = \frac{d\sin x}{\cos x} = \frac{d\sin x}{\sqrt{1-\sin^2 x}},\ dx = -\frac{d\cos x}{\sin x} = -$$

$$\frac{d\cos x}{\sqrt{1-\cos^2 x}},\ dx = \cos^2 x\,d\,tang\,x = \frac{d\,tang\,x}{\sec^2 x} =$$

$$\frac{d\,tang\,x}{1+tang^2 x},\ dx = -\sin^2 x\,d\cot x = -\frac{d\cot x}{cosec^2 x} =$$

$$-\frac{d\cot x}{1+\cot^2 x},\ dx = \frac{\cos x\,d\sec x}{tang\,x} = \frac{d\sec x}{\sec x\sqrt{\sec^2 x - 1}},$$

$$dx = -\frac{\sin x \, d\operatorname{cosec} x}{\cot x} = -\frac{d\operatorname{cosec} x}{\operatorname{cosec} x\sqrt{\operatorname{cosec}^2 x - 1}}.$$

Aequationes istae in hunc modum scribi possunt

$$d\,arc(\sin = z) = \frac{dz}{\sqrt{1-z^2}},\ d\,arc(\cos = z) = -\frac{dz}{\sqrt{1-z^2}},$$

$$d\,arc(\tan g = z) = \frac{dz}{1+z^2},\ d\,arc(\cot = z) = -\frac{dz}{1+z^2},$$

$$d\,arc(\sec = z) = \frac{dz}{z\sqrt{z^2-1}},\ d\,arc(\operatorname{cosec} = z) =$$

$$-\frac{dz}{z\sqrt{z^2-1}}.$$

12. Veniant quoque differentiandae functiones

$$z = y^u,\ z = y^{\frac{1}{u}},\ z = y^{u^v}, \ldots$$

1.°

$$L(z^2) = uL(y^2),\ \frac{dz}{z} = u\frac{dy}{y} + L(y)du,\ dz = dy^u =$$

$$y^{u-1}[udy + yL(y)du].$$

2.°

$$L(z^2) = \frac{1}{u}L(y^2),\ \frac{dz}{z} = \frac{u\frac{dy}{y} - L(y)du}{u^2},$$

$$dz = dy^{\frac{1}{u}} = \frac{y^{\frac{1-u}{u}}[udy - yL(y)du]}{u^2}.$$

3.° ,

$$L(z^z)=u^v L(y^z),\ \frac{dz}{z}=u^v\frac{dy}{y}+L(y)u^{v-1}[vdu+$$

$$uL(u)dv],\ dz=dy^{u^v}=y^{u^v}u^v[\frac{dy}{y}+$$

$$\frac{vL(y)}{u}du+L(y)L(u)dv].$$

et caet. . .

13. Quemadmodum ex $y=f(x)$ habuimus (6) $dy=f'(x)dx$, ita ex hac obtinebimus $ddy=f''(x)dxdx=f''(x)dx^2$, ex qua rursus $dddy=f'''(x)dx\,dx^2=f'''(x)dx^3$, atque ita porro; denotant f'', f''', ... novas functiones variabilis independentis x, cujus differentiale est constans et arbitraria quantitas. Itaque si compendii causa exhibentur ddy, $dddy$, ... per d^2y, d^3y, ... profluent

$$dy=f'(x)dx,\ d^2y=f''(x)dx^2,\ d^3y=f'''(x)dx^3,\ ...$$

$$d^ny=f^{(n)}(x)dx^n;\ \frac{dy}{dx}=f'(x),\ \frac{d^2y}{dx^2}=f''(x),$$

$$\frac{d^3y}{dx^3}=f'''(x),\ ...\ \frac{d^ny}{dx^n}=f^{(n)}(x).$$

Differentialia dy, d^2y, d^3y, ... d^ny itemque functiones derivatae $f'(x)$, $f''(x)$, $f'''(x)$, ... $f^{(n)}(x)$ dicuntur primi, secundi, tertii, ... n^{simi} ordinis respectu functionis primitivae $y=f(x)$. Eaedem $f'(x)$, $f''(x)$, ... solent etiam appellari *coefficientes differentiales*.

Exempla.

I.° $y=a^x$, $dy=L(a)a^xdx$, $d^2y=L^2(a)a^xdx^2$, ... $d^ny=L^n(a)a^xdx^n$. Si daretur $y=e^x$, esset $d^ny=e^xdx^n$.

II.° $y = ax^n$, $dy = nax^{n-1}dx$, $d^2y = n(n-1)ax^{n-2}dx^2$, $d^3y = n(n-1)(n-2)ax^{n-3}dx^3$, ... $d^ny = n(n-1)(n-2)(n-3)\ldots 2.1.adx^n$, $d^{n+1}y = 0$.

III.° $y = \mathrm{L}(x)$, $dy = \frac{dx}{x}$, $d^2y = -\frac{1.dx^2}{x^2}$,

$$d^3y = \frac{1.2dx^3}{x^3},\ d^4y = -\frac{1.2.3dx^4}{x^4},\ldots d^ny = \pm\frac{1.2.3.4\ldots(n-1)dx^n}{x^n};$$

valet signum superius si n est impar, inferius si par.

IV.° $y = \sin x$, $dy = \cos x dx = \sin\left(x + \frac{\pi}{2}\right)dx$,

$$d^2y = -\sin x\, dx^2 = \sin\left(x + \frac{2\pi}{2}\right)dx^2,$$

$$d^3y = -\cos x\, dx^3 = \sin\left(x + \frac{3\pi}{2}\right)dx^3,$$

$$d^4y = \sin x\, dx^4 = \sin\left(x + \frac{4\pi}{2}\right)dx^4,\ \ldots.$$

$$d^ny = \sin\left(x + \frac{n\pi}{2}\right)dx^n.$$

V.° $y = \cos x$, $dy = -\sin x\, dx = \cos\left(x + \frac{\pi}{2}\right)dx$,

$$d^2y = -\cos x\, dx^2 = \cos\left(x + \frac{2\pi}{2}\right)dx^2,$$

$$d^3y = \sin x\, dx^3 = \cos\left(x + \frac{3\pi}{2}\right)dx^3,$$

$$d^4y = \cos x\, dx^4 = \cos\left(x + \frac{4\pi}{2}\right) dx^4, \ldots.$$

$$d^n y = \cos\left(x + \frac{n\pi}{2}\right) dx^n.$$

14. Ex (h: 7) habemus quidem $dz = F'(y)dy$ similiter ac quoad $y = f(x)$; at cum dy (utpote $= f'(x)dx$) nequeat assumi ut quantitas constans, iccirco in eruendis aliis differentialibus functionis z habenda quoque erit ratio differentialium ipsius dy. Exsurget itaque (9)

$d^2z = F''(y)dy^2 + F'(y)d^2y$, $d^3z = F'''(y)dy^3 + 3F''(y)dy\, d^2y + F'(y)d^3y$, $d^4z = F^{IV}(y)dy^4 + 6F'''(y)dy^2d^2y + 4F''(y)dyd^3y + 3F''(y)\overline{d^2y}^2 + F'(y)d^4y$, et caet. . . .

Hinc

$$F'(y) = \frac{dz}{dy}, F''(y) = \frac{dyd^2z - dzd^2y}{dy^3} = \frac{1}{dy} d\left(\frac{dz}{dy}\right),$$

$$F'''(y) = \frac{dy(dyd^3z - dzd^3y) - 3d^2y(dyd^2z - dzd^2y)}{dy^5} =$$

$$\frac{1}{dy} d\left(\frac{dyd^2z - dzd^2y}{dy^3}\right), \quad F^{IV}(y) =$$

$$\frac{\begin{array}{l} dy^3d^4z - 6dy^2d^2yd^3z - 4dy^2d^2zd^3y \\ +15dy\overline{d^2y}^2 d^2z + 10dydzd^2yd^3y \\ -15dzd^2y^3 - dy^2dzd^4y \end{array}}{dy^7} =$$

$$\frac{1}{dy} d\left(\frac{dy(dyd^3z - dzd^3y) - 3d^2y(dyd^2z - dzd^2y)}{dy^5}\right),$$

et caet. . . .

15. Si variabilis y evaderet independens, foret $d^2y=0$, $d^3y=0$, $d^4y=0$, . . . ; unde

$$F'(y)=\frac{dz}{dy},\ F''(y)=\frac{d^2z}{dy^2},\ F'''(y)=\frac{d^3z}{dy^3},\ F^{IV}(y)=\frac{d^4z}{dy^4},\ldots$$

formulae prorsus similes illis, quas invenimus (13) quoad variabilem independentem x. Hinc vero colligimus, si habeantur functiones derivatae F', F'', F''', .., expressae per differentialia tum functionis primitivae $z=F(y)$, tum variabilis y, colligimus inquam fore F' eamdem sive y ponatur independens, sive non, caeteras autem esse alias in primo casu atque in secundo: poterit vero ab ipso primo casu ad secundum transiri substituendo $\frac{dyd^2z-dzd^2y}{dy^3}$,

$\frac{dy(dyd^3z-dzd^3y)-3d^2y(dyd^2z-dzd^2y)}{dy^5}$,... in locum

$$\frac{d^2z}{dy^2},\ \frac{d^3z}{dy^3},\ \ldots$$

16. Data expressione imaginaria variabili $p+q\sqrt{-1}$, si pro realibus quantitatibus p et q substituuntur respectivi limites, novam expressionem imaginariam inde resultantem voco (245 ex p. 1.^{a}) limitem illius datae. Hinc aptantes expressioni imaginariae

$$s=u+v\sqrt{-1}$$

jam traditas (6) de differentialibus, derivatisque functionibus notiones, habebimus

$$ds=du+dv\sqrt{-1},\ d^2s=d^2u+d^2v\sqrt{-1},\ \text{et caet.} \ldots$$

$$\frac{ds}{dx}=\frac{du}{dx}+\frac{dv}{dx}\sqrt{-1},\ \frac{d^2s}{dx^2}=\frac{d^2u}{dx^2}+\frac{d^2v}{dx^2}\sqrt{-1},$$

et caet. . . .

Quibus positis, erunt v. gr. (6. 7.° 8.° 6.° 5° : 11 : 7)

$$d[\cos(x\mathrm{L}(a)) + \sqrt{-1}\,\sin(x\mathrm{L}(a))] =$$

$$[\cos(x\mathrm{L}(a)) + \sqrt{-1}\sin(x\mathrm{L}(a))]\sqrt{-1}\,\mathrm{L}(a)dx\,,$$

$$d[\mathrm{L}(a^2 + x^2)^{\frac{1}{2}} + \sqrt{-1}\;arc(tang = \frac{x}{a})] =$$

$$\frac{x+a\sqrt{-1}}{x^2+a^2}dx = \frac{x+a\sqrt{-1}}{(x+a\sqrt{-1})(x-a\sqrt{-1})}dx =$$

$$\frac{dx}{x-a\sqrt{-1}} = \frac{\sqrt{-1}\,dx}{a+x\sqrt{-1}}\,,$$

$$d\left(\frac{e^x+e^{-x}}{2}\cos a + \frac{e^{-x}-e^x}{2}\sqrt{-1}\,\sin a\right) =$$

$$-\left(\frac{e^x+e^{-x}}{2}\sin a + \frac{e^x-e^{-x}}{2}\sqrt{-1}\,\cos a\right)\sqrt{-1}\,dx\,,$$

$$d\left(\frac{e^x+e^{-x}}{2}\sin a + \frac{e^x-e^{-x}}{2}\sqrt{-1}\,\cos a\right) =$$

$$\left(\frac{e^x+e^{-x}}{2}\cos a + \frac{e^{-x}-e^x}{2}\sqrt{-1}\,\sin a\right)\sqrt{-1}\,dx\,,$$

et caet. . . . ; ideoque (248 : g^{iv} ex p. 1.^{a} et 162. 1°. 5° : 168. 1° ex p. 2.^{a})

$$da^{x\sqrt{-1}} = a^{x\sqrt{-1}}\sqrt{-1}\,\mathrm{L}(a)dx\,,$$

$$d\mathrm{L}(a+x\sqrt{-1}) = \frac{d(a+x\sqrt{-1})}{a+x\sqrt{-1}}\,,\; d\cos(a+x\sqrt{-1}) =$$

$-\sin(a+x\sqrt{-1})d(a+x\sqrt{-1})$, $d\sin(a+x\sqrt{-1})=$

$\cos(a+x\sqrt{-1})d(a+x\sqrt{-1})$, et caet. . . . ;

prorsus ut quoad consimiles expressiones reales. Ad haec habemus (162. 6.° ex p. 2.[a])

$$dL(-x)=d[L(x)\pm(2m+1)\pi\sqrt{-1}]=dL(x)=\frac{dx}{x};$$

sive igitur sit $x>0$, sive <0, ejus logarithmus eodem donatur differentiali.

DE RELATIONE INTER FUNCTIONES UNIUS VARIABILIS ET RESPECTIVAS DERIVATAS.

17. Si crescit vel decrescit x ita, ut evadat major vel minor quam peculiaris valor x_n, quaeritur utrum in viciniis x_n una cum x crescat vel decrescat functio $y=f(x)$, quae in iisdem viciniis ponitur esse continua.

Quoniam (6)

$$lim.\frac{\Delta y}{\Delta x}=\frac{dy}{dx}=f'(x);$$

hinc 1.° si $f'(x_n)>0$; eodem certe afficientur signo Δy et Δx quum x est quam proxime x_n, ibique proinde aucta vel imminuta x crescet simul vel decrescet y: 2.° si $f'(x_n)<0$, diversis afficientur signis Δy et Δx dum x est quam proxime x_n, ibique propterea aucta x decrescet y, imminuta x crescet y.

18. Si binae functiones $f(x)$, $\varphi(x)$, et quae ab ipsis deducuntur $f'(x)$; $\varphi'(x)$ sunt continuae inter limites x_0, x_n, ac praeterea $\varphi(x)$ vel constanter crescit, vel constanter descrescit ab x_0 ad x_n, erit semper aliquis numerus $\varepsilon<1$ et >0 satisfaciens aequationi

$$\frac{f(x_n)-f(x_0)}{\varphi(x_n)-\varphi(x_0)}=\frac{f'(x_0+\varepsilon(x_n-x_0))}{\varphi'(x_0+\varepsilon(x_n-x_0))}\ldots(q).$$

Exhibeat M maximum, et m minimum omnium valorum quos per totum intervallum $x_n - x_0$ recipit fractio $\frac{f'(x)}{\varphi'(x)}$: quisque videt (83 ex p. 1.a) differentias

$$\frac{f'(x)}{\varphi'(x)}-m,\ \mathrm{M}-\frac{f'(x)}{\varphi'(x)}$$

fore ambas positivas: ergo (17)

$$\varphi'(x)\left(\frac{f'(x)}{\varphi'(x)}-m\right),\ \varphi'(x)\left(\mathrm{M}-\frac{f'(x)}{\varphi'(x)}\right),$$

seu

$$f'(x)-m\varphi'(x),\ \mathrm{M}\varphi'(x)-f'(x)$$

idem habebunt signum ab x_0 ad x_n. Atqui hae quantitates nihil sunt aliud nisi primi ordinis derivatae ex functionibus

$$f(x)-m\varphi(x),\ \mathrm{M}\varphi(x)-f(x):$$

igitur hujusmodi primitivae functiones crescent simul, aut decrescent ab x_0 ad x_n; ideoque

$$f(x_n)-m\varphi(x_n)-(f(x_0)-m\varphi(x_0)),$$

$$\mathrm{M}\varphi(x_n)-f(x_n)-(\mathrm{M}\varphi(x_0)-f(x_0)),$$

seu

$$f(x_n)-f(x_0)-m(\varphi(x_n)-\varphi(x_0)),$$

$$\mathrm{M}(\varphi(x_n)-\varphi(x_0))-(f(x_n)-f(x_0))$$

erunt simul vel positivae, vel negativae. Hinc autem facile intelligitur quod binae quantitates

$$\frac{f(x_n)-f(x_0)}{\varphi(x_n)-\varphi(x_0)}-m,\ \frac{f(x_n)-f(x_0)}{\varphi(x_n)-\varphi(x_0)}-\mathrm{M}$$

debent contrariis affici signis : unde sequitur fractionem

$$\frac{f(x_n)-f(x_0)}{\varphi(x_n)-\varphi(x_0)}$$

fore inter M et m. Jam vero valores omnes, qui circumscribuntur limitibus istis, repraesentantur per secundum membrum aequationis (q), assumpto ε inter 0 et 1 : erit igitur aliquis numerus $\varepsilon < 1$ et > 0, qui eidem (q) satisfaciet.

19. In aequatione (q) substitue x pro x_0, et $x+\Delta x$ pro x_n : ea fiet

$$\frac{f(x+\Delta x)-f(x)}{\varphi(x+\Delta x)-\varphi(x)}=\frac{f'(x+\varepsilon.\Delta x)}{\varphi'(x+\varepsilon.\Delta x)}\ldots(q');$$

quae formula valet quotiescumque et $f(x)$, $\varphi(x)$, $f'(x)$, $\varphi'(x)$ sunt continuae inter x ac $x+\Delta x$, et $\varphi(x)$ vel constanter crescit vel constanter decrescit ab x_0 ad x_n.

20. Functiones $f(x)$, $\varphi(x)$ ponantur evanescere quando loco x substituitur quidam peculiaris valor x_n; erit (19)

$$\frac{f(x_n+\Delta x)}{\varphi(x_n+\Delta x)}=\frac{f'(x_n+\varepsilon.\Delta x)}{\varphi'(x_n+\varepsilon.\Delta x)}\ldots(q'').$$

Jam si

$$f(x),\ f'(x),\ f''(x),\ldots f^{(m-1)}(x),\ f^{(m)}(x)$$
$$\varphi(x),\ \varphi'(x),\ \varphi''(x),\ldots \varphi^{(m-1)}(x),\ \varphi^{(m)}(x)$$

permaneant continuae ab $x=x_n$ ad $x=x_n+\Delta x$, exemptisque $f^{(m)}(x)$, $\varphi^{(m)}(x)$ evanescant caeterae quando fit $x=x_n$, atque insuper

$$\varphi(x),\ \varphi'(x),\ \varphi''(x),\ldots\varphi^{(m-1)}(x)$$

aut constanter crescant, aut constanter decrescant inter praefatos limites x_n, $x_n+\Delta x$, poterit manifeste

continuari aequatio (q'') in hunc modum

$$\left.\begin{array}{l}\dfrac{f(x_n+\Delta x)}{\varphi(x_n+\Delta x)}=\dfrac{f'(x_n+\varepsilon.\Delta x)}{\varphi'(x_n+\varepsilon.\Delta x)}=\\ \dfrac{f''(x_n+\varepsilon_1.\Delta x)}{\varphi''(x_n+\varepsilon_1.\Delta x)}=\ldots=\dfrac{f^{(m)}(x_n+\varepsilon_{m-1}\Delta x)}{\varphi^{(m)}(x_n+\varepsilon_{m-1}\Delta x)};\end{array}\right\}(q''')$$

denotant ε, ε_1, ε_2, ε_3, ... ε_{m-1} ejusmodi numeros, ut $\varepsilon > \varepsilon_1 > \varepsilon_2 > \varepsilon_3 > \ldots$, ac singuli < 1 et > 0. Retento itaque ε ad quemvis ex hisce numeris generatim exprimendum, et praetermissis in (q''') fractionibus intermediis, erit

$$\frac{f(x_n+\Delta x)}{\varphi(x_n+\Delta x)}=\frac{f^{(m)}(x_n+\varepsilon.\Delta x)}{\varphi^{(m)}(x_n+\varepsilon.\Delta x)} \quad \ldots \quad (q^{\text{IV}})$$

21. Assumpta $\varphi(x)=(x-x_n)^m$, erit $\varphi^{(m)}(x)=1\,.\,2\,.\,3\,\ldots\,m$, $\varphi(x_n+\Delta x)=\Delta x^m$, $\varphi^{(m)}(x_n+\varepsilon\Delta x)=1\,.\,2\,.\,3\,\ldots\,m$; ideoque vertetur ($q^{\text{IV}}$) in

$$\frac{f(x_n+\Delta x)}{\Delta x^m}=\frac{f^{(m)}(x_n+\varepsilon.\Delta x)}{1\,.\,2\,.\,3\,\ldots m} \quad \ldots \quad (q^{\text{V}}).$$

At si una cum $f^{(m)}(x)$, $\varphi^{(m)}(x)$ etiam $f(x)$ permaneret aliqua quando fit $x=x_n$, prodiret

$$\frac{f(x_n+\Delta x)-f(x_n)}{\Delta x^m}=\frac{f^{(m)}(x_n+\varepsilon.\Delta x)}{1\,.\,2\,.\,3\ldots m};$$

unde

$$f(x_n+\Delta x)-f(x_n)=$$

$$\frac{\Delta x^m}{1\,.\,2\,.\,3\ldots m}f^{(m)}(x_n+\varepsilon\,.\,\Delta x)\ldots(q^{\text{VI}}).$$

22. In formula (q^{V}) facto $x_n=0$, substituto insuper δ loco Δx, et adhibita littera F pro f, exsurget

$$F(\delta)=\frac{\delta^m}{1.2.3\ldots m}F^{(m)}(\varepsilon\delta)\ldots\ldots(q^{\text{vii}}),$$

quae subsistet quotiescumque functiones

$$F(\delta)\ ,\ F'(\delta)\ ,\ F''(\delta)\ ,\ldots\ldots F^{(m)}(\delta)$$

erunt continuae, initium ducendo ab $\delta=0$, et exempta $F^{(m)}(\delta)$ caeterae evanescent, ipsa δ evanescente.

23. Detur polynomium $f(z+\delta)-f(z)-\frac{\delta}{1}f'(z)-$

$$\frac{\delta^2}{1.2}f''(z)-\frac{\delta^3}{1.2.3}f'''(z)-\ldots-\frac{\delta^{m-1}}{1.2.3\ldots(m-1)}f^{(m-1)}(z)$$

spectandum uti functio quantitatis δ: liquet ejus functiones derivatas (ponimus esse continuas una cum ipso polynomio) usque ad ordinem m^{simum} fore

$$f'(z+\delta)-f'(z)-\frac{\delta}{1}f''(z)-\frac{\delta^2}{1.2}f'''(z)-\ldots\ldots$$

$$-\frac{\delta^{m-2}}{1.2.3\ldots(m-2)}f^{(m-1)}(z)\ ,\ f''(z+\delta)-$$

$$f''(z)-\frac{\delta}{1}f'''(z)-\ldots-\frac{\delta^{m-3}}{1.2.3\ldots(m-3)}f^{(m-1)}(z)\ ,$$

et caet. ....

$$f^{(m-1)}(z+\delta)-f^{(m-1)}(z)\ ,\ f^{(m)}(z+\delta).$$

Jam si m^{simam} excipias, facta $\delta=0$ caeterae omnes derivatae una cum functione primitiva evanescunt: igitur (22 : q^{vii})

$$f(z+\delta)=f(z)+\frac{\delta}{1}f'(z)+\frac{\delta^2}{1.2}f''(z)+\frac{\delta^3}{1.2.3}f'''(z)+\ldots$$

$$+\frac{\delta^{m-1}}{1.2.3\ldots(m-1)}f^{(m-1)}(z)+$$

$$+\frac{\delta^m}{1.2.3\ldots m}f^{(m)}(z+\varepsilon\delta)\ldots(q^{\mathrm{VIII}}).$$

Mutata z in zero et δ in z,

$$f(z)=f(0)+\frac{z}{1}f'(0)+\frac{z^2}{1.2}f''(0)+\frac{z^3}{1.2.3}f'''(0)+\ldots$$

$$+\frac{z^{m-1}}{1.2.3\ldots(m-1)}f^{(m-1)}(0)+\frac{z^m}{1.2.3\ldots m}f^{(m)}(\varepsilon z)\ldots(q^{\mathrm{IX}}).$$

Exempla.

I.° Ponatur $f(z)=e^z$; erit (13) $f'(z)=e^z$, $f''(z)=e^z$, ... $f^{(m)}(z)=e^z$: hinc $f(0)=1$, $f'(0)=1$, $f''(0)=1$, ... $f^{(m)}(\varepsilon z)=e^{\varepsilon z}$; ac proinde

$$e^z=1+\frac{z}{1}+\frac{z^2}{1.2}+\frac{z^3}{1.2.3}+\ldots$$

$$+\frac{z^{m-1}}{1.2.3\ldots(m-1)}+\frac{z^m e^{\varepsilon z}}{1.2.3\ldots m}.$$

II.° Sit $f(z)=\mathrm{L}(1+z)$; erit (13) $f'(z)=\frac{1}{1+z}$,

$$f''(z)=-\frac{1}{(1+z)^2},\quad f'''(z)=\frac{1.2}{(1+z)^3},$$

$$f^{\mathrm{IV}}(z)=-\frac{1.2.3}{(1+z)^4},\ldots f^{(m)}(z)=\pm\frac{1.2.3\ldots(m-1)}{(1+z)^m};$$

unde $f(0)=0$, $f'(0)=1$, $f''(0)=-1$, $f'''(0)=1.2$, $f^{\mathrm{IV}}(0)=-1.2.3$, ... $f^{(m-1)}(0)=\mp 1.2.3\ldots(m-2)$,

$$f^{(m)}(\varepsilon z)=\pm\frac{1.2.3\ldots(m-1)}{(1+\varepsilon z)^m};$$ et consequenter

$$L(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \ldots\ldots$$

$$\mp \frac{z^{m-1}}{m-1} \pm \frac{z^m}{m(1+\varepsilon z)^m}:$$

valebit signum superius ubi m est impar, inferius ubi par.

III.° Sumatur $f(z+\delta)=L(z+\delta)$; erit $f(z)=L(z)$,

$$f'(z)=\frac{1}{z},\ f''(z)=-\frac{1}{z^2},\ f'''(z)=\frac{1\,.\,2}{z^3},\ f^{\mathrm{iv}}(z)=$$

$$-\frac{1.2.3}{z^4},\ \ldots f^{(m-1)}(z)=\mp\frac{1.\ 2.\ 3\ldots\ldots(m-2)}{z^{m-1}},$$

$$f^{(m)}(z+\varepsilon\delta)=\pm\frac{1.\ 2.\ 3\ldots(m-1)}{(z+\varepsilon\delta)^m};$$

ideoque

$$L(z+\delta)=L(z)+\frac{\delta}{z}-\frac{\delta^2}{2z^2}+\frac{\delta^3}{3z^3}-\ldots\ldots$$

$$\mp\frac{\delta^{m-1}}{(m-1)z^{m-1}}\pm\frac{\delta^m}{m(z+\varepsilon\delta)^m}.$$

24. Denotante δ_1 peculiarem valorem quantitatis δ, substitue $\delta-\delta_1$ pro δ et $z+\delta_1$ pro z in (q^{VIII}. 23); habebis

$$\left.\begin{array}{l} f(z+\delta)=f(z+\delta_1)+\dfrac{\delta-\delta_1}{1}f'(z+\delta_1)+ \\ \dfrac{(\delta-\delta_1)^2}{1.2}f''(z+\delta_1)+\ldots+\dfrac{(\delta-\delta_1)^{m-1}}{1.2\ldots(m-1)}f^{(m-1)}(z+\delta_1)+ \\ \dfrac{(\delta-\delta_1)^m}{1.2\ldots m}f^{(m)}[z+\delta_1+\varepsilon(\delta-\delta_1)]. \end{array}\right\}\ (q^{\mathrm{X}})$$

Mutata z in zero, δ in z, et δ_1 in z_1, prodibit

$$\left.\begin{array}{l} f(z)=f(z_1)+\frac{z-z_1}{1}f'(z_1)+\frac{(z-z_1)^2}{1.2}f''(z_1)+\dots \\ \frac{(z-z_1)^{m-1}}{1.2\dots(m-1)}f^{(m-1)}(z_1)+\frac{(z-z_1)^m}{1.2\dots m}f^{(m)}[z_1+\varepsilon(z-z_1)]. \end{array}\right\}(q^{\text{XI}})$$

Exemplum.

Sit $f(z)=z^a$; erit (13) $f'(z)=az^{a-1}$, $f''(z)=a(a-1)z^{a-2}$, ... $f^{(m)}(z)=a(a-1)(a-2)\dots(a-m+1)z^{a-m}$: proinde

$$z^a=z_1{}^a+\frac{z-z_1}{1}az_1{}^{a-1}+\frac{(z-z_1)^2}{1.2}a(a-1)z_1{}^{a-2}+\dots$$

$$+\frac{(z-z_1)^{m-1}}{1.2\dots(m-1)}a(a-1)\dots(a-m+2)z_1{}^{a-m+1}+$$

$$\frac{(z-z_1)^m}{1.2\dots m}a(a-1)\dots(a-m+1)[z_1+\varepsilon(z-z_1)]^{a-m}.$$

25. Pone

$$\left.\begin{array}{l} f(z+\delta)-f(z+\delta_1)-\frac{\delta-\delta_1}{1}f'(z+\delta_1)- \\ \frac{(\delta-\delta_1)^2}{1.2}f''(z+\delta_1)-\dots-\frac{(\delta-\delta_1)^{m-1}}{1.2\dots(m-1)}f^{(m-1)}(z+\delta_1)=\mathrm{f}(\delta_1), \end{array}\right\}(q^{\text{XII}})$$

et sume differentialia quoad δ_1: pervenies ad

$$\mathrm{f}'(\delta_1)=-\frac{(\delta-\delta_1)^{m-1}}{1.2\dots(m-1)}f^{(m)}(z+\delta_1);$$

unde

$$\mathrm{f}'(\delta_1+\varepsilon'\delta)=-\frac{(\delta-\delta_1-\varepsilon'\delta)^{m-1}}{1.2\dots(m-1)}f^{(m)}(z+\delta_1+\varepsilon'\delta):$$

exprimit ε' numerum aliquem >0 et <1.

Sed (21 . q^{VI})

$$f(\delta_1+\delta)-f(\delta_1)=\delta\,.\,f'(\delta_1+\varepsilon'\delta)\,;$$

igitur

$$f(\delta_1+\delta)-f(\delta_1)=-\frac{\delta(\delta-\delta_1-\varepsilon'\delta)^{m-1}}{1.2\ldots(m-1)}f^{(m)}(z+\delta_1+\varepsilon'\delta)\,;$$

et consequenter

$$f(\delta)-f(0)=-\frac{\delta(\delta-\varepsilon'\delta)^{m-1}}{1.2\ldots(m-1)}f^{(m)}(z+\varepsilon'\delta).$$

Jamvero facta $\delta=\delta_1$ habemus ex (q^{XII})

$$f(\delta)=0\,,$$

et facta $\delta_1=0$,

$$f(0)=f(z+\delta)-f(z)-\frac{\delta}{1}f'(z)-\frac{\delta^2}{1.2}f''(z)-\ldots$$
$$-\frac{\delta^{m-1}}{1.2\ldots(m-1)}f^{(m-1)}(z)\,;$$

ergo (23 . q^{VIII})

$$\left.\begin{aligned}f(0)=\frac{\delta^m}{1.2\ldots m}f^{(m)}(z+\varepsilon\delta)=\\ \frac{\delta^m(1-\varepsilon')^{m-1}}{1.2\ldots(m-1)}f^{(m)}(z+\varepsilon'\delta).\end{aligned}\right\}(q^{\text{XIII}})$$

Mutata z in zero, et δ in z, proveniet

$$\frac{z^m}{1.2\ldots m}f^{(m)}(\varepsilon z)=\frac{z^m(1-\varepsilon')^{m-1}}{1.2\ldots(m-1)}f^{(m)}(\varepsilon' z)\ldots(q^{\text{XIV}}).$$

DE RATIONE DETERMINANDI VALORES FUNCTIONUM UNIUS VARIABILIS SESE EXHIBENTIUM SUB QUIBUSDAM FORMIS INDETERMINATIS.

26. Vergente Δx ad *lim.* $= 0$, formula (q^{III} : 20) dat in limite

$$\frac{f(x_n)}{\varphi(x_n)} = \frac{f'(x_n)}{\varphi'(x_n)} = \frac{f''(x_n)}{\varphi''(x_n)} = \ldots = \frac{f^{(m)}(x_n)}{\varphi^{(m)}(x_n)} \quad \ldots (g)$$

cujus ope determinantur valores illarum fractionum, quae formam recipiunt indeterminatam $\frac{0}{0}$ ubi in ipsis loco x subrogetur peculiaris valor x_n.

Exempla.

I.° Sint $f(x) = \mathrm{L}(1+x)$, $\varphi(x) = x$, quae, facto $x = x_n = 1$, praebent

$$\frac{f(x_n)}{\varphi(x_n)} = \frac{\mathrm{L}(1)}{0} = \frac{0}{0}.$$

Habemus $f'(x) = \frac{1}{1+x}$, $\varphi'(x) = 1$, unde $f'(x_n) = 1$, $\varphi'(x_n) = 1$; et consequenter

$$\frac{f(x_n)}{\varphi(x_n)} = \frac{\mathrm{L}(1)}{0} = \frac{0}{0} = 1.$$

II.° Sume $f(x) = x + (ax - a - 1)\, x^{a+1}$, $\varphi(x) = (x-1)^2$, quae, facto $x = x_n = 1$, suppeditant

$$\frac{f(x_n)}{\varphi(x_n)} = \frac{0}{0}:$$

item cum exsistant $f'(x) = 1 + (a+1)(ax - a - 1)x^a + ax^{a+1}$, $\varphi'(x) = 2(x-1)$, erit adhuc

$$\frac{f''(x_n)}{\varphi'(x_n)} = \frac{0}{0} ;$$

at quoniam $f''(x) = a(a+1)(ax-a-1)x^{a-1} + 2a(a+1)x^a$, $\varphi''(x)=2$, ideo

$$\frac{f(x_n)}{\varphi(x_n)} = \frac{f'(x_n)}{\varphi'(x_n)} = \frac{0}{0} = \frac{f''(x_n)}{\varphi''(x_n)} = \frac{a(a+1)}{2} .$$

27. Si $f(x_n) = \infty$, $\varphi(x_n) = \infty$, ut fractio sese exhibeat sub forma indeterminata $\frac{\infty}{\infty}$, cum habeamus

$$\frac{\frac{1}{f(x_n)}}{\frac{1}{\varphi(x_n)}} = \frac{0}{0} ,$$

cumque

$$\frac{d\frac{1}{f(x)}}{dx} = -\frac{f'(x)}{f^2(x)} , \quad \frac{d\frac{1}{\varphi(x)}}{dx} = -\frac{\varphi'(x)}{\varphi^2(x)} ,$$

iccirco (26)

$$\frac{\frac{1}{f(x_n)}}{\frac{1}{\varphi(x_n)}} = \frac{\frac{f'(x_n)}{f^2(x_n)}}{\frac{\varphi'(x_n)}{\varphi^2(x_n)}} , \quad \text{unde} \quad \frac{f(x_n)}{\varphi(x_n)} = \frac{f'(x_n)}{\varphi'(x_n)} .$$

Si praeter $f(x_n)$, $\varphi(x_n)$ etiam $f'(x_n)$, $\varphi'(x_n)$ prodeunt $= \infty$, simili modo ostendetur (26) fore

$$\frac{f'(x_n)}{\varphi'(x_n)} = \frac{f''(x_n)}{\varphi''(x_n)} ;$$

atque ita porro. Formula nimirum (g : 26) suppeditat

quoque valores fractionum sese exhibentium sub forma indeterminata $\frac{\infty}{\infty}$.

Ad haec:

$$f(x)\cdot\varphi(x)=\frac{f(x)}{\frac{1}{\varphi(x)}}=\frac{\varphi(x)}{\frac{1}{f(x)}};$$

per eamdem videlicet (g) obtinebitur valor facti $f(x_n)\cdot\varphi(x_n)$, quotiescumque binarum $f(x_n)$, $\varphi(x_n)$ altera existente $=0$, altera $=\infty$, ipsum $f(x_n)\cdot\varphi(x_n)$ sese exhibet sub forma indeterminata $0\cdot\infty$.

Exempla.

I.° Sint $f(x)=\mathrm{L}\left(\frac{1}{x}\right)$, $\varphi(x)=\cot x$; erunt $f'(x)=-\frac{1}{x}$, $\varphi'(x)=-\frac{1}{\sin^2 x}$: proinde (129. 2.° ex p. 2^a.) quoad $x=0$

$$\frac{\mathrm{L}\left(\frac{1}{x}\right)}{\cot x}=\frac{\infty}{\infty}=\frac{\sin^2 x}{x}=\frac{\sin x}{x}\sin x=\sin x=0.$$

II°. Pone $f(x)=x^c$, $\varphi(x)=e^x$, et m immediate $>c$; habebis (13) $f^{(m)}(x)=c(c-1)\ldots(c-m+1)x^{c-m}$, $\varphi^{(m)}(x)=e^x$: propterea quoad $x=\infty$

$$\frac{x^c}{e^x}=\frac{\infty}{\infty}=\frac{c(c-1)\ldots(c-m+1)}{x^{m-c}e^x}=0.$$

Hinc facto $x=\frac{1}{\beta}$, erit quoad $\beta=0$

$$\frac{e^{-\frac{1}{\beta}}}{\beta^c}=0;$$

utcumque magnus de caetero accipiatur numerus finitus atque positivus c : habita nimirum β pro basi, erit $e^{-\frac{1}{\beta}}$ consideranda (250 ex p. 1.^{a}) veluti quantitas infinitesima ordinis ∞.

III.° Sume $f(x) = \mathrm{L}(x)$, $\varphi(x) = x^c$, unde

$$f'(x) = \frac{1}{x}, \quad \frac{d\frac{1}{\varphi(x)}}{dx} = -\frac{c}{x^{c+1}} \text{; erit quoad } x = 0$$

$$\mathrm{L}(x) \cdot x^c = -\infty \cdot 0 = \frac{\frac{1}{x}}{-\frac{c}{x^{c+1}}} = -\frac{x^c}{c} = 0 \,;$$

prorsus ut in p. 1.^{a} n.° 241. 3.°

28. Est (208 ex p. 1.^{a})

$$[f(x)]^{\varphi(x)} = e^{\varphi(x) \cdot \mathrm{L}[f(x)]} \,;$$

hinc si peculiari valori x_n respondet $f(x_n) = 0$ et $\varphi(x_n) = 0$, vel $f(x_n) = \infty$ et $\varphi(x_n) = 0$, vel etiam $f(x_n) = 1$ et $\varphi(x_n) = \infty$, ad inveniendum valorem functionis sese exhibentis sub prima, secunda, vel tertia e formis indeterminatis

$$0^0 \,,\; \infty^0 \,,\; 1^\infty ,$$

satis erit definire (26 : 27) valorem rationis

$$\frac{\mathrm{L}[f(x_n)]}{[\varphi(x_n)]^{-1}} \,.$$

Exempla

1°. Sit $f(x) = \varphi(x) = x$: quoniam $\dfrac{d\mathrm{L}(x)}{dx} = \dfrac{1}{x}$ et

$\frac{dx^{-1}}{dx} = -\frac{1}{x^2}$, ideo (27) quoad $x = x_n = 0$ erit $\frac{L(x)}{x^{-1}} = -x = 0$, ac proinde

$$x^x = 0^0 = e^0 = 1.$$

II.°. Positis $f(x) = x$, $\varphi(x) = \frac{1}{1-x}$, cum habeamus $\frac{dL(x)}{dx} = \frac{1}{x}$, $d\left(\frac{1}{1-x}\right)^{-1} = -1$, propterea (26) quoad $x = x_n = 1$ erit $\frac{L(x)}{\left(\frac{1}{1-x}\right)^{-1}} = \frac{1}{-1.x} = -1$; et consequenter

$$x^{\frac{1}{1-x}} = 1^\infty = e^{-1} = \frac{1}{e}.$$

29. Notetur illud: si $f(\beta)$ est quantitas infinitesima, cujus basis $=\beta$, ordo $=c$, ac 1.° ponatur c numerus integer, simulque (250 ex p. 1a.)

$$lim. \frac{f(\beta)}{\beta^c} = 0,$$

erit

$$\frac{f(\beta)}{\beta^{c+1}}$$

primus terminorum qui in progressione geometrica

$$f(\beta), \frac{f(\beta)}{\beta}, \frac{f(\beta)}{\beta^2}, \frac{f(\beta)}{\beta^3}, \ldots$$

non evanescunt una cum β: quod si 2.° haud prodeat

$$\lim.\frac{f(\beta)}{\beta^c}=0,$$

utcumque se habeat c, expresso per m numero integro vel $=c$, vel immediate $>c$, erit

$$\frac{f(\beta)}{\beta^m}$$

primus terminorum qui in progressione illa non evanescunt una cum β. Jamvero ii termini aequivalent (22: q^{VII}) respective terminis seriei

$$f(\beta),\ f'(\varepsilon\beta),\ \frac{f''(\varepsilon\beta)}{2},\ \frac{f'''(\varepsilon\beta)}{2.3},\ldots$$

Ergo $f^{(c+1)}(\beta)$ in 1°. casu, et $f^{(m)}(\beta)$ in 2°. erit prima, quae ex functionibus

$$f(\beta),\ f'(\beta),\ f''(\beta),\ f'''(\beta),\ldots$$

desinit evanescere una cum β.

Hinc quia in ordine v. gr. ad

$$e^{-\frac{1}{\beta}}$$

est (27. II°.) $c=\infty$, nullus ex terminis

$$e^{-\frac{1}{\beta}},\ \frac{de^{-\frac{1}{\beta}}}{d\beta},\ \frac{d^2e^{-\frac{1}{\beta}}}{d\beta^2},\ \frac{d^3e^{-\frac{1}{\beta}}}{d\beta^3},\ldots.$$

desinet evanescere una cum β.

DE MAXIMIS, MINIMISQUE VALORIBUS FUNCTIONIS CONTINUAE $f(x)$.

30. Concipiatur x augeri parum admodum ultra peculiarem valorem x_n, vel imminui infra eumdem x_n:

si respondens functio $f(x)$ in utroque casu continue decrescat, erit valor $f(x_n)$ maximus inter eos, quos recipit functio $f(x)$; at si $f(x)$ continue crescat tam in primo quam in secundo casu, erit $f(x_n)$ minimus (131. 3°. ex p. 1^a.) inter valores, quos obtinet $f(x)$. Hinc (17) statim atque x ex $<$ inceperit esse $>x_n$, vel ex $>$ inceperit esse $<x_n$, derivata $f'(x)$ fiet negativa, quae prius erat positiva, et contra fiet positiva, quae prius erat negativa, si quidem $f(x_n)$ esse debeat maximus, vel minimus inter vicinos valores $f(x)$. Quare (131. 1°. ex p. 1^a.) facta $x=x_n$, evanescet $f'(x)$ si et ipsa est continua similiter ac $f(x)$; sin minus, evadet infinita. Jam non pluribus opus est ut intelligamus valores x_n, quibus respondet maxima vel minima $f(x_n)$, quaerendos esse inter radices aequationum

$$f'(x)=0\,,\quad \frac{1}{f'(x)}=0.$$

Illud quoque facile intelligitur (1°): functio $f(x_n)$ est maxima quotiescumque

$f'(x)<0$ quoad $x>x_n$, et $f'(x)>0$ quoad $x<x_n$;

est autem minima quotiescumque

$f'(x)>0$ quoad $x>x_n$, et $f'(x)<0$ quoad $x<x_n$.

Quod si $f'(x)$ maneat aut constanter positiva, aut constanter negativa dum x versatur in viciniis x_n, certe $f(x_n)$ neque maxima erit, neque minima.

Exempla

I.° Sit $f(x)=x^5-5x^4+5x^3+1$, ideoque $f'(x)=5x^4-20x^3+15x^2$; aequatio $f'(x)=0$ praebet $x_n=0$, $x_n=1$, $x_n=3$: denotante σ quantitatem infinitesimam sive positivam sive negativam, substituatur $x_n+\sigma$ loco x in $f'(x)$; sumpto $x_n=0$, erit

$$f''(x)=5\sigma^4-20\sigma^3+15\sigma^2,$$

quae cum quamproxime x_n perseveret (251 : 252 ex p. 1.^{a}) constanter positiva, primus valor $x_n=0$ neque maximam dabit $f(x)$, neque minimam; sumpto $x_n=1$, erit

$$f'(x)=(1+\sigma)^2\ (5\sigma^2-10\sigma),$$

quae cum quam proxime x_n evadat negativa si σ fuerit positiva idest si $x>x_n$, evadat autem positiva si σ fuerit negativa idest si $x<x_n$, suppeditabit proinde secundus valor $x_n=1$ functionem $f(x)$ maximam; sumpto denique $x_n=3$, erit

$$f'(x)=(3+\sigma)^2\ (5\sigma^2+10\sigma),$$

quae cum in viciniis x_n sit positiva si $\sigma>0$ nimirum si $x>x_n$, sit vero negativa si $\sigma<0$ videlicet si $x<x_n$, dabit iccirco tertius valor $x_n=3$ functionem $f(x)$ minimam.

II°. Proponatur consideranda functio $f(x)=\dfrac{\mathrm{L}(x)}{x}$,

unde $f'(x)=\dfrac{1-\mathrm{L}(x)}{x^2}$: ex $f'(x)=0$ eruimus $x_n=e$;

est autem $\dfrac{1-\mathrm{L}(e+\sigma)}{(e+\sigma)^2}$ negativa si $\sigma>0$, positiva si $\sigma<0$; nimirum $f'(x)<0$ quoad $x>x_n$, $f'(x)>0$ quoad $x<x_n$: valor igitur $x_n=e$ praebet functionem $f(x)$ maximam.

31. Formula (q^{VI} : 21) aliam nobis regulam suppeditat, qua possimus dignoscere utrum datae radici x_n aequationis $f'(x)=0$ respondeat maxima, vel minima functio $f(x_n)$. Secundum namque membrum illius formulae in viciniis x_n aut manet constanter <0, aut constanter >0, sive Δx accipiatur positiva sive negativa, aut modo fit >0, modo <0, prout mutatur signum ipsius Δx: in primo casu erit $f(x_n)>f(x_n\pm\Delta x)$, ideoque maxima; in secundo $f(x_n)<f(x_n\pm\Delta x)$, et

consequenter minima; in tertio $f(x_n)$ neque constanter major, neque constanter minor quam $f(x_n \pm \Delta x)$, ac proinde neque maxima, neque minima. Permanebit autem illud secundum membrum aut constanter < 0, aut constanter > 0 tunc solum, quum exponens m quantitatis Δx est par; siquidem $f^{(m)}(x_n \pm \varepsilon\Delta x)$ in viciniis x_n eodem afficitur signo ac $f^{(m)}(x_n)$. Ergo si ex derivatis

$$f''(x_n),\ f'''(x_n),\ f^{\text{IV}}(x_n), \ldots$$

quae non evanescit prima, est ordinis imparis, certe $f(x_n)$ neque maxima erit, neque minima; quod si fuerit ordinis paris, derivatae $f^{(m)}(x_n) < 0$ respondebit maxima $f(x_n)$, derivatae $f^{(m)}(x_n) > 0$ respondebit minima $f(x_n)$.

Sic in 1°. exemplo (30) habemus

$$f''(x) = 20x^3 - 60x^2 + 30x,\ f'''(x) = 60x^2 - 120x + 30,$$

et caet. . . .

Jamvero $f''(x_n) = 0$ et $f'''(x_n) = 30$ quoad $x_n = 0$, $f''(x_n) = -10 < 0$ quoad $x_n = 1$, $f''(x_n) = 90 > 0$ quoad $x_n = 3$: hinc rursus e valoribus x_n primus neque maximam neque minimam dabit functionem, secundus dabit maximam, tertius minimam.

Sic etiam in 2.° exemplo (30) assequimur

$$f''(x) = \frac{2L(x) - 3}{x^3}, \text{ et caet.} \ldots;$$

unde $f''(x_n) = \dfrac{2L(e) - 3}{e^3} = -\dfrac{1}{e^3} < 0$: valor nimirum $x_n = e$ praebet functionem maximam.

32. Possunt binae functiones $f(z+\delta)$, $f(z)$ ita considerari, ut altera coalescat (23) e serie

$$f(z),\ \frac{\delta}{1}f'(z),\ \frac{\delta^2}{1.2}f''(z),\ \frac{\delta^3}{1.2.3}f'''(z),\ \text{et caet.}\ \ldots\ (b)$$

et residuo

$$\frac{\delta^m}{1.2\ldots m}f^{(m)}(z+\varepsilon\delta)\ \ldots\ (b'),$$

altera e serie

$$f(0),\ \frac{z}{1}f'(0),\ \frac{z^2}{1.2}f''(0),\ \frac{z^3}{1.2.3}f'''(0),\ \text{et caet.}\ \ldots\ (b'')$$

et residuo

$$\frac{z^m}{1.2\ldots m}f^{(m)}(\varepsilon z)\ \ldots\ (b''').$$

Jam si, crescente m indefinite, vergant residua (b'), (b''') ad $lim. = 0$, certe serierum (b), (b'') altera verget ad $lim. = f(z+\delta)$, altera ad $lim. = f(z)$: in ea igitur qua sumus hypothesi valebunt (233 ex p. 1.^{a}) binae

$$\left.\begin{aligned} f(z+\delta) &= f(z)+\frac{\delta}{1}f'(z)+\frac{\delta^2}{1.2}f''(z)+\frac{\delta^3}{1.2.3}f'''(z)+\ldots,\\ f(z) &= f(0)+\frac{z}{1}f'(0)+\frac{z^2}{1.2}f''(0)+\frac{z^3}{1.2.3}f'''(0)+\ldots;\end{aligned}\right\}\ (b^{\text{IV}})$$

quarum prima est *formula Taylori*, secunda *Mac-Laurini.*

33. Constat ex dictis in p. 1^a. n.° 236, si m crescit indefinite, fractiones

$$\frac{\delta^m}{1.2\ldots m},\ \frac{z^m}{1.2\ldots m}$$

ad $lim. = 0$ indefinite accessuras, utcumque de caetero sumantur valores finiti δ, z. Quamobrem residua (b'), (b''') vergent ad $lim. = 0$ quotiescumque functiones $f^{(m)}(z + \varepsilon\delta)$, $f^{(m)}(\varepsilon z)$ sunt ejusmodi, ut non crescant indefinite una cum m: licet autem functiones istae, aucto m in infinitum, quemcumque datum limitem praetergrediantur, fieri tamen potest (27. II.°) ut residua illa maneant adhuc infinitesima.

34. In primo exemplo (23) $f^{(m)}(\varepsilon z) = e^z$; permanet videlicet $f^{(m)}(\varepsilon z)$, utcumque crescat m: iccirco $lim. (b''') = 0$; et

$$e^z = 1 + z + \frac{z^2}{2} + \frac{z^3}{2.3} + \frac{z^4}{2.3.4} + \ldots.$$

quoad omnes valores finitos z.

In secundo exemplo (23)

$$f^{(m)}(\varepsilon z) = \pm \frac{1.2.3 \ldots (m-1)}{(1 + \varepsilon z)^m};$$

et quia, aucto m indefinite, crescit in infinitum (236 ex p. 1.^{a}) valor quantitatis

$$\pm \frac{1.2.3 \ldots (m-1)}{(1 + \varepsilon z)^m},$$

idipsum ergo dicendum de $f^{(m)}(\varepsilon z)$: at cum habeamus (25 : q^{XIV})

$$(b''') = \pm \frac{z^m(1-\varepsilon')^{m-1}}{(1+\varepsilon' z)^m} = \pm \frac{1}{1-\varepsilon'}\left(\frac{z-\varepsilon' z}{1+\varepsilon' z}\right)^m,$$

profecto ab $z > -1$ ad $z = 1$ exsistet $lim. (b''') = 0$; et consequenter intra limites illos erit

$$L(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \ldots.$$

Recole p. 1^{am} : 241 : 242.

35. Ponatur 1.°

$$f(z+\delta)=a_0+a_1\delta+a_2\delta^2+a_3\delta^3+a_4\delta^4+\ldots;$$

sumptis derivatis quoad δ, erunt

$$f'(z+\delta)=a_1+2a_2\delta+3a_3\delta^2+4a_4\delta^3+\ldots,$$
$$f''(z+\delta)=2a_2+2.3a_3\delta+3.4a_4\delta^2+\ldots,$$
$$f'''(z+\delta)=2.3a_3+2.3.4a_4\delta+\ldots,$$

et caet. . . . ;

et facto $\delta=0$,

$$a_0=f(z),\ a_1=f'(z),\ a_2=\tfrac{1}{2}f''(z),\ a_3=\frac{1}{2.3}f'''(z),$$

et caet. . . .

Ponatur 2°.

$$f(z)=b_0+b_1z+b_2z^2+b_3z^3+b_4z^4+\ldots;$$

exsurgent

$$f'(z)=b_1+2b_2z+3b_3z^2+4b_4z^3+\ldots,$$
$$f''(z)=2b_2+2.3b_3z+3.4b_4z^2+\ldots,$$
$$f'''(z)=2.3b_3+2.3.4b_4z+\ldots,$$

et caet. . . . ;

et facto $z=0$,

$$b_0=f(0),\ b_1=f'(0),\ b_2=\tfrac{1}{2}f''(0),\ b_3=\frac{1}{2.3}f'''(0),$$

et caet. . . .

Hinc ubi $f(z+\delta)$ fuerit summa cujuspiam seriei convergentis *ordinatae per ascendentes potentias quantitatis* δ, et $f(z)$ summa cujuspiam seriei convergentis ordinatae per potentias ascendentes quantitatis z, certe istiusmodi serierum altera recidet in (b), altera in (b'').

36. Permanente functione $e^{-\frac{1}{z}}$, evanescunt (29) omnes respondentes termini seriei (b''). Hinc si ponitur v. gr.

$$f(z) = e^z + e^{-\frac{1}{z}},$$

respondens series (b'') erit quidem convergens (236 ex p. 1.^{a}), sed ejus summa consistet in primo dumtaxat termino e^z, non in toto binomio $e^z + e^{-\frac{1}{z}}$: fieri videlicet potest ut formula Mac-Laurini suppeditet evolutionem functionis in seriem convergentem quin tamen ejusmodi seriei summa recidat in functionem ipsam.

DE DIFFERENTIATIONE FUNCTIONUM PLURES COMPLECTENTIUM VARIABILES.

37. Sit $\mu = f(x, y, z, \ldots)$ functio plurium variabilium independentium x, y, z, ... Porro vel istarum una, vel duae, aut plures, aut omnes sua recipient incrementa Δx, Δy, Δz, ... In hoc ultimo casu

$$\Delta\mu = f(x+\Delta x, y+\Delta y, z+\Delta z, \ldots) - f(x, y, z, \ldots)$$

dicitur *totalis* differentia functionis μ: in aliis casibus differentiae vocantur *partiales*. Si v. gr. crescat vel sola x, vel sola y, erunt

$$f(x+\Delta x, y, z, \ldots) - f(x, y, z, \ldots),$$
$$f(x, y+\Delta y, z, \ldots) - f(x, y, z, \ldots)$$

differentiae partiales functionis μ, altera quoad x, altera quoad y; quae differentiae designantur etiam per $\Delta_x\mu$, $\Delta_y\mu$, ...

38. Si functio μ est continua (5) quoad singulas $x, y, z, \ldots$ erit quoque continua quoad omnes. Assumptis enim Δx, Δy, Δz, ... infinitesimis, quantitates

$f(x+\Delta x, y, z, \ldots) - f(x, y, z, \ldots)$,

$f(x+\Delta x, y+\Delta y, z, \ldots) - f(x + \Delta x, y, z, \ldots)$, $f(x+\Delta x, y + \Delta y, z+\Delta z, \ldots) - f(x+\Delta x, y+\Delta y, z, \ldots)$; et caet. ...

ex hypothesi vergent singulae ad limitem $=0$, prima simul cum Δx, secunda cum Δy, tertia cum Δz, ...; ad eumdem ergo *lim.*$=0$ verget ipsarum summa

$$f(x+\Delta x, y+\Delta y, z+\Delta z, \ldots) - f(x, y, z \ldots),$$

seu $\Delta\mu$ simul cum omnibus Δx, Δy, Δz, ...; ac proinde et caet. ...

39. Pone (6) $\Delta x = i\beta$, $\Delta y = i'\beta$, $\Delta z = i''\beta$, ..., unde

$$lim. \frac{\Delta x}{\beta} = lim. i, \; lim. \frac{\Delta y}{\beta} = lim\, i', \; lim. \frac{\Delta z}{\beta} = lim.\, i'', \ldots :$$

fac insuper ut functio μ exsistat continua (38) intra quaedam confinia variabilibus $x, y, z, \ldots$ assignata; etsi $\Delta\mu$ intra confinia illa vergit ad *lim.*$=0$, attamen ratio $\frac{\Delta\mu}{\beta}$ poterit ad alium limitem vergere sive $>$, sive <0: limites rationum

$$\frac{\Delta x}{\beta}, \; \frac{\Delta y}{\beta}, \; \frac{\Delta z}{\beta}, \ldots$$

appellamus differentialia variabilium independentium $x, y, z, \ldots$; limitem vero rationis

$$\frac{\Delta\mu}{\beta}$$

totale primi ordinis differentiale functionis μ; designamusque per dx, dy, dz, ... $d\mu$, ut scribi possit

$$dx = lim. \frac{\Delta x}{\beta} = lim. i, \ dy = lim. \frac{\Delta y}{\beta} = lim. i', $$

$$dz = lim. \frac{\Delta z}{\beta} = lim. i'', \ \ldots\ldots d\mu = lim. \frac{\Delta \mu}{\beta}$$

40. Si quaevis una e quantitatibus x, y, z, ... spectetur uti variabilis et habeantur caeterae pro constantibus, poterunt differentialia functionis μ eodem manifeste modo determinari ac differentialia functionum quae ab unica pendent variabili. Ejusmodi differentialia dicuntur partialia, ipsaque sic exhibebimus ut

$$d_x\mu, \ d^2{}_x\mu, \ \ldots \ d_y\mu, \ d^2{}_y\mu, \ \ldots$$

denotent differentialia functionis μ, primi, secundi, ... ordinis quoad x, quoad y, ...

41. Ad partiales functiones derivatas quod pertinet, eae poterunt sic exprimi ut per

$$\frac{d_x\mu}{dx}, \ \frac{d^2{}_x\mu}{dx^2}, \ldots \frac{d_y\mu}{dy}, \ \frac{d^2{}_y\mu}{dy^2}, \ldots$$

vel per

$$f'_x(x, y, z, \ldots), \ f''_x(x, y, z, \ldots), \ldots$$

$$f'_y(x, y, z, \ldots), \ f''_y(x, y, z, \ldots), \ldots$$

designentur functiones primi, secundi, ... ordinis, derivatae ex $\mu = f(x, y, z, \ldots)$ quoad x, quoad y, ... Plerumque tamen in his derivatis functionibus exprimendis detrahemus, compendii causa, litterae d signa x, y, z, ..., et pro

$$\frac{d_x\mu}{dx}, \ \frac{d^2{}_x\mu}{dx^2}, \ldots \frac{d_y\mu}{dy}, \ \frac{d^2{}_y\mu}{dy^2}, \ldots$$

adhibebimus

$$\frac{d\mu}{dx}, \frac{d^2\mu}{dx^2}, \dots \frac{d\mu}{dy}, \frac{d^2\mu}{d^2y}, \dots .$$

Hinc vero

$$d_x\mu = \frac{d_x\mu}{dx}dx = f'_x(x, y, z, \dots) dx = \frac{d\mu}{dx} dx ,$$

$$d^2{}_x\mu = \frac{d^2{}_x\mu}{dx^2}dx^2 = f''_x(x, y, z, \dots) dx^2 = \frac{d^2\mu}{dx^2}dx^2 ,$$

et caet. . . .

$$d_y\mu = \frac{d_y\mu}{dy}dy = f'_y(x, y, z, \dots) dy = \frac{d\mu}{dy} dy ,$$

$$d^2{}_y\mu = \frac{d^2{}_y\mu}{dy^2}dy^2 = f''_y(x, y, z, \dots) dy^2 = \frac{d^2\mu}{dy^2}dy^2 ,$$

et caet. . . .

42. Potest etiam functio μ differentiari successive quoad binas, ternas, . . . variabiles v. gr. quoad x, y, quoad x, y, z; et caet. . . Id genus partialia secundi, tertii, ... ordinis differentialia designamus per

$$d_y d_x\mu , \ d_z d_y d_x\mu , \dots$$

43. Sive functio μ prius differentietur v. gr. quoad x deinde quoad y, sive prius quoad y deinde quoad x, idem in utroque casu obtinebitur differentiale. Nam functionis μ differentias (37) capienti, primo quoad x, y, postea quoad y, x, protinus apparebit fore

$$\frac{\Delta_y\left(\frac{\Delta_x\mu}{\beta}\right)}{\beta} = \frac{\Delta_x\left(\frac{\Delta_y\mu}{\beta}\right)}{\beta} ;$$

unde, facto ad limites gradu,

$$d_y d_x \mu = d_x d_y \mu.$$

Simili modo

$$d_y d_z \mu = d_z d_y \mu,\ d_x d_z \mu = d_z d_x \mu,\ \ldots$$

Hinc manifeste profluunt

$d_x d_y d_z \ldots \mu = d_x d_z d_y \ldots \mu = d_z d_x d_y \ldots \mu =$
$d_z d_y d_x \ldots \mu = \ldots,\ d^2{}_x d_y d_z \ldots \mu = d_x d_y d_x d_z \ldots \mu =$
$d_y d^2{}_x d_z \ldots \mu = \ldots,\ d^2{}_x d^3{}_y d_z \ldots \mu =$
$d_y d^2{}_x d^2{}_y d_z \ldots \mu = d_y d_x d^2{}_y d_x d_z \ldots \mu =$
$d^3{}_y d^2{}_x d_z \ldots \mu = \ldots$, et caet. ...

Ex quibus pronum est universim concludere istiusmodi differentialia proventura semper eadem, quocumque demum ordine perficiantur successivae differentiationes respectu variabilium $x, y, z, \ldots$

44. Quoniam differentialia omnia $dx, dy, dz, \ldots$ variabilium independentium $x, y, z, \ldots$ sunt (6 : 39) constantes et arbitrariae quantitates, sequitur (13 : 40)

$$d^2{}_x d_y d_z \mu,\ d^3{}_x d^2{}_y \mu,\ d^4{}_y d^3{}_z \mu,\ \ldots$$

nihil fore aliud nisi novas ipsarum $x, y, z, \ldots$ functiones respective multiplicatas per

$$dx^2 dy dz,\ dx^3 dy^2,\ dy^4 dz^3,\ \ldots$$

ut generatim habeamus

$$d^m{}_x d^n{}_y d^h{}_z \ldots \mu = \varphi(x, y, z, \ldots) dx^m dy^n dz^h \ldots;$$

unde partialis derivata functio ordinis $m+n+h+\ldots$

$$\varphi(x, y, z, \ldots) = \frac{d^m{}_x d^n{}_y d^h{}_z \ldots \mu}{dx^m dy^n dz^h},$$

et dectractis hic quoque (41) litterae d, compendii gratia, signis $x, y, z, \ldots$

$$\varphi(x, y, z, \ldots\ldots) = \frac{d^{m+n+h+\ldots}\mu}{dx^m dy^n dz^h \ldots}.$$

45. Totale functionis μ differentiale $d\mu$ eruitur ex partialibus $d_x\mu$, $d_y\mu$, $d_z\mu$, ... Designent enim ε_1, ε_2, ε_3, .. valores numericos > 0 et < 1 : erunt (21 . q^{vi} : 41)

$$f(x+\Delta x, y, z, \ldots) - f(x, y, z, \ldots) =$$

$$\Delta x f'_x(x+\varepsilon_1\Delta x, y, z, \ldots),$$

$$f(x+\Delta x, y+\Delta y, z, \ldots) - f(x+\Delta x, y, z, \ldots) =$$

$$\Delta y f'_y(x+\Delta x, y+\varepsilon_2\Delta y, z, \ldots),$$

$$f(x+\Delta x, y+\Delta y, z+\Delta z, \ldots) -$$

$$f(x+\Delta x, y+\Delta y, z, \ldots) =$$

$$\Delta z f'_z(x+\Delta x, y+\Delta y, z+\varepsilon_3\Delta z, \ldots),$$

et caet. ..., quarum summa praebet

$$\Delta\mu = \Delta x f'_x(x+\varepsilon_1\Delta x, y, z, \ldots) + \Delta y f'_y(x+\Delta x, y+\varepsilon_2\Delta y,$$

$$z \ldots) + \Delta z f'_z(x+\Delta x, y+\Delta y, z+\varepsilon_3\Delta z, \ldots) + \ldots (h).$$

Hinc (39)

$$\frac{\Delta\mu}{\beta} = i f'_x(x+\varepsilon_1 i\beta, y, z, \ldots) + i' f'_y(x+i\beta,$$

$$y+\varepsilon_2 i'\beta, z, \ldots) + i'' f'_z(x+i\beta, y+i'\beta, z+\varepsilon_3 i''\beta, \ldots) + \ldots;$$

et facto ad limites gradu,

$$d\mu = df(x, y, z, \ldots) = f'_x(x, y, z, \ldots)\,dx +$$

$$f'_y(x, y, z, \ldots)\,dy + f'_z(x, y, z, \ldots)\,dz + \ldots;$$

quod poterit etiam exprimi per hunc triplicem modum (44)

$$\left.\begin{aligned} d\mu &= d_x\mu + d_y\mu + d_z\mu + \ldots, \\ d\mu &= \frac{d_x\mu}{dx}dx + \frac{d_y\mu}{dy}dy + \frac{d_z\mu}{dz}dz + \ldots, \\ d\mu &= \frac{d\mu}{dx}dx + \frac{d\mu}{dy}dy + \frac{d\mu}{dz}dz + \ldots \end{aligned}\right\} (h').$$

Totale nimirum differentiale functionis μ coalescentis ex pluribus variabilibus $x, y, z, \ldots$ independentibus obtinebitur colligendo in summam differentialia partialia $d_x\mu$, $d_y\mu$, $d_z\mu$, ...

Exempla.

I.° Sit $\mu = \dfrac{x^3 - x\mathrm{L}(y)}{\sin z}$: prodibunt

$d_x\mu = \dfrac{3x^2 - \mathrm{L}(y)}{\sin z}dx$, $d_y\mu = -\dfrac{x}{y \sin z}dy$,

$d_z\mu = -\dfrac{x^3 - x\mathrm{L}(y)}{\sin^2 z}\cos z\, dz$; unde

$$d\mu = \frac{3x^2 - \mathrm{L}(y)}{\sin z}dx - \frac{x}{y \sin z}dy - \frac{x^3 - x\mathrm{L}(y)}{\sin^2 z}\cos z\, dz.$$

II.° Detur $\mu = (z - \sqrt{y})^x$; erunt

$d_x\mu = (z - \sqrt{y})^x \mathrm{L}(z - \sqrt{y})dx$, $d_y\mu = -\dfrac{x(z - \sqrt{y})^{x-1}}{2\sqrt{y}}dy$,

$d_z\mu = x(z - \sqrt{y})^{x-1}dz$; ac proinde

$$d\mu = \mathrm{L}(z - \sqrt{y})(z - \sqrt{y})^x dx - \frac{x}{2\sqrt{y}}(z - \sqrt{y})^{x-1}dy + x(z - \sqrt{y})^{x-1}dz.$$

46. Totalia secundi, tertii, . . . ordinis differentialia $dd\mu$, $ddd\mu$, . . . seu $d^2\mu$, $d^3\mu$, . . . facile determinantur. Sic v. gr. (45. h')

$$d^2\mu = dd\mu = d(d_x\mu + d_y\mu + d_z\mu + \ldots) =$$
$$d_x(d_x\mu + d_y\mu + d_z\mu + \ldots) + d_y(d_x\mu + d_y\mu + d_z\mu + \ldots) +$$
$$d_z(d_x\mu + d_y\mu + d_z\mu + \ldots) + \ldots;$$

unde (43)

$$d^2\mu = d^2{}_x\mu + d^2{}_y\mu + d^2{}_z\mu + \ldots + 2d_x d_y\mu + 2d_x d_z\mu + \ldots$$
$$+ 2d_y d_z\mu + \ldots,$$

seu (41 : 44)

$$d^2\mu = \frac{d^2\mu}{dx^2}dx^2 + \frac{d^2\mu}{dy^2}dy^2 + \frac{d^2\mu}{dz^2}dz^2 + \ldots +$$
$$2\frac{d^2\mu}{dxdy}dxdy + 2\frac{d^2\mu}{dxdz}dxdz + \ldots +$$
$$2\frac{d^2\mu}{dydz}dydz + \ldots.$$

Rursus

$$d^3\mu = dd^2\mu = d(d^2{}_x\mu + d^2{}_y\mu + d^2{}_z\mu + \ldots$$
$$+ 2d_x d_y\mu + 2d_x d_z\mu + \ldots + 2d_y d_z\mu + \ldots);$$

ideoque (45 : 43)

$$d^3\mu = d^3{}_x\mu + d^3{}_y\mu + d^3{}_z\mu + \ldots +$$
$$3d_x d^2{}_y\mu + 3d_x d^2{}_z\mu + \ldots + 3d_y d^2{}_x\mu +$$
$$3d_y d^2{}_z\mu + \ldots + 3d_z d^2{}_x\mu + 3d_z d^2{}_y\mu + \ldots +$$
$$6d_x d_y d_z\mu + \ldots,$$ seu (41 : 44)

$$d^3\mu = \frac{d^3\mu}{dx^3}dx^3 + \frac{d^3\mu}{dy^3}dy^3 + \frac{d^3\mu}{dz^3}dz^3 + \ldots +$$

$$3\frac{d^3\mu}{dxdy^2}dxdy^2 + 3\frac{d^3\mu}{dxdz^2}dxdz^2 + \ldots +$$

$$3\frac{d^3\mu}{dydx^2}dydx^2 + 3\frac{d^3\mu}{dydz^2}dydz^2 + \ldots +$$

$$3\frac{d^3\mu}{dzdx^2}dzdx^2 + 3\frac{d^3\mu}{dzdy^2}dzdy^2 + \ldots +$$

$$6\frac{d^3\mu}{dxdydz}dxdydz + \ldots .$$

Simili modo eruuntur $d^4\mu$, $d^5\mu$, ...

47. Constitutas differentiandi regulas attendenti patebit eosdem proventuros valores $d\mu$, $d^2\mu$, $d^3\mu$, ... sive differentietur functio μ, habitis x, y, z, . . . pro independentibus, sive differentietur

$$f(x+\theta dx\,,\, y+\theta dy\,,\, z+\theta dz\,, \ldots)$$

ita, ut ex quantitatibus

$$x+\theta dx\,,\, y+\theta dy\,,\, z+\theta dz\,, \ldots$$

unaquaeque pro unico habeatur termino; tum occurrentibus

$$d(x+\theta dx)\,,\, d(y+\theta dy)\,,\, d(z+\theta dz)\,, \ldots .$$

et peragantur istae differentiationes respectu solius θ, et dividantur per $d\theta$ differentialia inde orta; hisque peractis fiat demum ubique $\theta = 0$. Atqui peculiaris ejusmodi operandi modus in id recidit, ut habita sola θ pro variabili, facta insuper

$$f(x+\theta dx\,,\, y+\theta dy\,,\, z+\theta dz\,, \ldots) = \varphi(\theta) \ldots (h''),$$

determinatisque functionibus derivatis

$$\varphi'(\theta)\,,\, \varphi''(\theta)\,,\, \varphi'''(\theta)\,, \ldots$$

fiat dein ubique $\theta=0$: posita igitur aequatione (h''), et consequenter

$$\left.\begin{array}{l}\mu = f(x, y, z, \ldots) = \varphi(0), \\ \text{erunt quoque} \\ d\mu = \varphi'(0), \; d^2\mu = \varphi''(0), \; d^3\mu = \varphi'''(0), \ldots\end{array}\right\} (h''')$$

48. Veniat nunc consideranda functio

$$U = F(u, v, s, \ldots)$$

coalescens ex functionibus u, v, s, ... variabilium independentium x, y, z, ... Quo pacto aequationem (h : 45) obtinuimus, eodem assequemur

$$F(u+\Delta u, v+\Delta v, s+\Delta s, \ldots) - F(u, v, s, \ldots) =$$

$$\Delta U = \Delta u F'_u(u + \varepsilon_1 \Delta u, v, s, \ldots) + \Delta v F'_v(u + \Delta u,$$

$$v + \varepsilon_2 \Delta v, s, \ldots) + \Delta s F'_s(u + \Delta u, v + \Delta v, s + \varepsilon_3 \Delta s, \ldots) + \ldots;$$

tum adhibita divisione per β et facto ad limites gradu, proveniet (39) totale functionis U differentiale

$$dU = F'_u(u, v, s, \ldots) du + F'_v(u, v, s, \ldots) dv +$$

$$F'_s(u, v, s, \ldots) ds + \ldots,$$

quod per hunc triplicem modum poterit quoque exprimi (41)

$$\left.\begin{array}{l}dU = d_u U + d_v U + d_s U + \ldots, \\ dU = \dfrac{d_u U}{du} du + \dfrac{d_v U}{dv} dv + \dfrac{d_s U}{ds} ds + \ldots, \\ dU = \dfrac{dU}{du} du + \dfrac{dU}{dv} dv + \dfrac{dU}{ds} ds \ldots\end{array}\right\} (h^{\text{IV}})$$

Differentialia du, dv, ds, ... determinantur per (h':45)

Exempla.

I.° Sit $U = u + v + s + \ldots$; erit

$$d_u U = du,\ d_v U = dv,\ d_s U = ds, \ldots$$

ideoque

$$dU = du + dv + ds + \ldots$$

II.° $U = uvs \ldots$; $d_u U = vs \ldots du$,

$$d_v U = us \ldots dv,\ d_s U = uv \ldots ds, \ldots$$

Quare

$$dU = vs \ldots du + us \ldots dv + uv \ldots ds + \ldots$$

III.° $U = \frac{u}{v}$; $d_u U = \frac{du}{v}$, $d_v U = -\frac{udv}{v^2}$;

$$dU = \frac{du}{v} - \frac{udv}{v^2} = \frac{vdu - udv}{v^2}.$$

IV.° $U = u^v$; $d_u U = vu^{v-1}du$, $d_v U = L(u)u^v dv$;

$$dU = vu^{v-1}du + L(u)u^v dv.$$

Itaque regulae jam traditae (8 : 9 . . .) pro differentiatione functionum, quae ab unica pendent variabili, valent etiam pro functionibus plures independentes variabiles complectentibus.

49. Quod ad differentialia pertinet altiorum ordinum d^2U, d^3U, . . . functionis U, habemus v. gr.

$$d^2U = ddU = d\left(\frac{dU}{du}du + \frac{dU}{dv}dv + \frac{dU}{ds}ds + \ldots\right).$$

Jam si differentietur $\frac{dU}{du}du$ successive quoad u, v, s, . . . eademque instauretur operatio in caeteris ter-

minis $\frac{dU}{dv}dv$, $\frac{dU}{ds}ds$, ..., cum prodeant

$$d_u\left(\frac{dU}{du}du\right)=\frac{d^2U}{du^2}du^2+\frac{dU}{du}d^2u\,,$$

$$d_v\left(\frac{dU}{du}du\right)=\frac{d^2U}{dudv}dudv\,,\ldots\; d_u\left(\frac{dU}{dv}dv\right)=$$

$$\frac{d^2U}{dudv}dudv\,,\; d_v\left(\frac{dU}{dv}dv\right)=\frac{d^2U}{dv^2}dv^2+\frac{dU}{dv}d^2v\,,\ldots,$$

erit

$$d^2U=\frac{d^2U}{du^2}du^2+\frac{d^2U}{dv^2}dv^2+\frac{d^2U}{ds^2}ds^2+\ldots+$$

$$2\frac{d^2U}{dudv}dudv+2\frac{d^2U}{duds}duds+\ldots+$$

$$2\frac{d^2U}{dvds}dvds+\ldots+\frac{dU}{du}d^2u+\frac{dU}{dv}d^2v+$$

$$\frac{dU}{ds}d^2s+\ldots\;(h^v).$$

Simili modo pervenietur ad d^3U, d^4U, ...

DE AEQUATIONIBUS DIFFERENTIALIBUS.

50. Functio $U=F(u, v, s, \ldots)$ ponatur vel constantem obtinere valorem C, vel fieri $=0$, ut sit vel $U=C$, vel $U=0$; in utroque casu valebunt aequationes differentiales

$$dU=0\,,\; d^2U=0\,,\; d^3U=0\,,\ldots$$

In ea vero qua sumus hypothesi ex quantitatibus u,

v, s, ... una quaevis v. gr. u habenda erit pro functione caeterarum v, s, ... ; iccirco

du, d^2u, ... fient (48. h^{IV} : 49. h^{V})

$$\frac{du}{dv}dv + \frac{du}{ds}ds + \ldots, \frac{d^2u}{dv^2}dv^2 + \frac{d^2u}{ds^2}ds^2 + \ldots +$$

$$2\frac{d^2u}{dvds}dvds + \ldots + \frac{du}{dv}d^2v + \frac{du}{ds}d^2s + \ldots,$$

et caet.

Si variabiles v, s, ... evadunt independentes, evanescent ii termini, in quos ingrediuntur factores d^2v, d^3v, ... d^2s, d^3s, ...

51. Data v. gr. $\mu = f(x, y, z, \ldots) = 0$, unde $d\mu = 0$, seu (45. h')

$$\frac{d\mu}{dx}dx + \frac{d\mu}{dy}dy + \frac{d\mu}{dz}dz + \ldots = 0 \ldots (i),$$

si spectatur z uti functio implicita caeterarum x, y, ... independentium, cum in casu habeamus

$$dz = \frac{dz}{dx}dx + \frac{dz}{dy}dy + \ldots,$$

adhibita substitutione in (i), ea fiet

$$\left(\frac{d\mu}{dx} + \frac{d\mu}{dz} \cdot \frac{dz}{dx}\right)dx + \left(\frac{d\mu}{dy} + \frac{d\mu}{dz} \cdot \frac{dz}{dy}\right)dy + \ldots$$

$$= 0 \ldots (i'):$$

item (46)

$$d^2z = \frac{d^2z}{dx^2}dx^2 + \frac{d^2z}{dy^2}dy^2 + \ldots + 2\frac{d^2z}{dxdy}dxdy + \ldots$$

Quocirca aequatio differentialis $d^2\mu = 0$ seu (46 : 49)

$$\left.\begin{array}{l}\frac{d^2\mu}{dx^2}dx^2+\frac{d^2\mu}{dy^2}dy^2+\frac{d^2\mu}{dz^2}dz^2+\ldots+\\ 2\frac{d^2\mu}{dxdy}dxdy+2\frac{d^2\mu}{dxdz}dxdz+\ldots+\\ 2\frac{d^2\mu}{dydz}dydz+\ldots+\frac{d\mu}{dz}d^2z\end{array}\right\}=0$$

vertetur in

$$\left.\begin{array}{l}\left(\frac{d^2\mu}{dx^2}+\frac{d^2\mu}{dz^2}\left(\frac{dz}{dx}\right)^2+2\frac{d^2\mu}{dxdz}\cdot\frac{dz}{dx}+\right.\\ \left.\frac{d\mu}{dz}\cdot\frac{d^2z}{dx^2}\right)dx^2+2\left(\frac{dz}{dx}\cdot\frac{dz}{dy}\cdot\frac{d^2\mu}{dz^2}+\right.\\ \frac{d^2\mu}{dxdy}+\frac{d^2\mu}{dxdz}\cdot\frac{dz}{dy}+\frac{d^2\mu}{dydz}\cdot\frac{dz}{dx}+\\ \left.\frac{d\mu}{dz}\cdot\frac{d^2\mu}{dxdy}\right)dxdy+\left(\frac{d^2\mu}{dy^2}+\right.\\ \frac{d^2\mu}{dz^2}\left(\frac{dz}{dy}\right)^2+2\frac{d^2\mu}{dydz}\cdot\frac{dz}{dy}+\\ \left.\frac{d\mu}{dz}\cdot\frac{d^2z}{dy^2}\right)dy^2+\ldots\end{array}\right\}=0\ldots(i'')$$

et caet.....

52. Quoniam in (i', i'' et caet. .... 51) habentur x, y, ... et consequenter dx, dy, ... pro quantitatibus invicem non dependentibus, erunt

$$\left.\begin{array}{l}\frac{d\mu}{dx}+\frac{d\mu}{dz}\cdot\frac{dz}{dx}=0,\\ \frac{d\mu}{dy}+\frac{d\mu}{dz}\cdot\frac{dz}{dy}=0,\end{array}\right\}(i_1)$$

et caet....,

$$\left.\begin{array}{l}\left.\begin{array}{r}\frac{d^2\mu}{dx^2}+\frac{d^2\mu}{dz^2}\left(\frac{dz}{dx}\right)^2+2\frac{d^2\mu}{dxdz}\cdot\frac{dz}{dx}+\\ \frac{d\mu}{dz}\cdot\frac{d^2z}{dx^2}\,,\end{array}\right\}=0\\ \left.\begin{array}{r}\frac{dz}{dx}\cdot\frac{dz}{dy}\cdot\frac{d^2\mu}{dz^2}+\frac{d^2\mu}{dxdy}+\frac{d^2\mu}{dxdz}\cdot\frac{dz}{dy}+\\ \frac{d^2\mu}{dydz}\cdot\frac{dz}{dx}+\frac{d\mu}{dz}\cdot\frac{d^2z}{dxdy}\,,\end{array}\right\}=0\\ \left.\begin{array}{r}\frac{d^2\mu}{dy^2}+\frac{d^2\mu}{dz^2}\left(\frac{dz}{dy}\right)^2+2\frac{d^2\mu}{dydz}\cdot\frac{dz}{dy}+\\ \frac{d\mu}{dz}\cdot\frac{d^2z}{dy^2}\,,\end{array}\right\}=0\end{array}\right\}(i_2)$$

et caet. . . .

si quidem iisdem (i' , i'' , . . .) locus universim esse debeat. Differentiales aequationes (i_1 , i_2 , . . .) et quae ex earum combinatione utcumque promanant, vocantur partiales.

53. Istiusmodi aequationes adhiberi possunt ad eliminandas quantitates constantes quae in datam aequationem ingrediuntur, uti videre est in subjecto exemplo

Detur aequatio

$$z^2+ay^2+b(x+y)^2+c=0\,;$$

inde facile derivantur

$$b(x+y)+z\frac{dz}{dx}=0\,,\ ay+b(x+y)+z\frac{dz}{dy}=0\,,$$

ex quibus et ex data eruetur

$$z^2-z\left(x\frac{dz}{dx}+y\frac{dz}{dy}\right)+c=0\,,$$

ubi constantes a , b minime apparent.

Illud vero facile intelligitur : si proponatur aequatio ternas complectens variabiles x, y, z, cum ea suppeditet binas aequationes (i_1) primi ordinis, tres (i_2) secundi, quatuor tertii, ... $n+1$ n^{simi}, ac proinde differentialium ejusmodi aequationum totalem numerum (202 ex p. 1.^{a})

$$2+3+4+5+\ldots+(n+1)=\frac{(n+1)(n+2)}{1 \cdot 2}-1,$$

iccirco poterit inde et ex data elici aequatio ordinis n^{simi}, cui desint $\frac{(n+1)(n+2)}{1 \cdot 2}-1$ quantitates constantes.

Item (223 : 224 ex p. 1.^{a}) data aequatione inter quaternas variabiles, ea praebebit differentiales aequationes tres (i_1) primi ordinis, sex (i_2) secundi, decem tertii, quindecim quarti, ... $\frac{(n+1)(n+1)}{2}$ n^{simi}, et consequenter id genus aequationum totalem numerum

$$\frac{(n+1)(n+2)(n+3)}{1 \cdot 2 \cdot 3}-1.$$

Quocirca poterit inde et ex data obtineri partialis, ordinis n^{simi}, differentialis aequatio sine

$$\frac{(n+1)(n+2)(n+3)}{1 \cdot 2 \cdot 3}-1 \text{ constantibus.}$$

Universim data aequatione inter m variabiles, licebit ex ea et ex derivatis partialibus consimilem, ordinis n^{simi}, aequationem eruere absque

$$\frac{(n+1)(n+2)(n+3)\ldots(n+m-1)}{1 \cdot 2 \cdot 3 \ldots (m+1)}-1$$

constantibus in ipsam datam ingredientibus.

54. Iisdem differentialibus aequationibus conceditur eliminare indeterminatas functiones, si quas amplectitur proposita aequatio.

Exempla.

I.° Sit

$$\mu = f[\varphi(\nu),\ x,\ y,\ z] = 0:$$

designat ν datam variabilium independentium x, y functionem, et φ functionem omnino arbitrariam: facta $\varphi(\nu) = r$, cum habeamus

$$dr = \frac{dr}{dx}dx + \frac{dr}{dy}dy,\quad dz = \frac{dz}{dx}dx + \frac{dz}{dy}dy,$$

erit

$$\left.\begin{array}{l}\left(\dfrac{d\mu}{dr}\cdot\dfrac{dr}{dx} + \dfrac{d\mu}{dx} + \dfrac{d\mu}{dz}\cdot\dfrac{dz}{dx}\right)dx + \\ \left(\dfrac{d\mu}{dr}\cdot\dfrac{dr}{dy} + \dfrac{d\mu}{dy} + \dfrac{d\mu}{dz}\cdot\dfrac{dz}{dy}\right)dy,\end{array}\right\} = 0,$$

unde

$$\frac{d\mu}{dr}\cdot\frac{dr}{dx} + \frac{d\mu}{dx} + \frac{d\mu}{dz}\cdot\frac{dz}{dx} = 0,$$

$$\frac{d\mu}{dr}\cdot\frac{dr}{dy} + \frac{d\mu}{dy} + \frac{d\mu}{dz}\cdot\frac{dz}{dy} = 0.$$

Est autem

$$\frac{dr}{dx} = \varphi'(\nu)\frac{d\nu}{dx},\quad \frac{dr}{dy} = \varphi'(\nu)\frac{d\nu}{dy};$$

hinc

$$\varphi'(\nu)\frac{d\mu}{dr}\cdot\frac{d\nu}{dx}+\frac{d\mu}{dx}+\frac{d\mu}{dz}\cdot\frac{dz}{dx}=0,$$

$$\varphi'(\nu)\frac{d\mu}{dr}\cdot\frac{d\nu}{dy}+\frac{d\mu}{dy}+\frac{d\mu}{dz}\cdot\frac{dz}{dy}=0.$$

Jam ex hisce binis aequationibus et ex primitiva $\mu=0$ eliminando $\varphi(\nu)$ ac $\frac{d\mu}{dr}\varphi'(\nu)$ perveniemus ad partialem, primi ordinis, aequationem differentialem arbitrariis privatam functionibus.

II.° $z=x\varphi(\nu)+\psi(\nu)$; exprimunt φ, ψ binas functiones indeterminatas, ν functionem datam variabilium independentium x, y.

Partialia aequationis propositae differentialia capientes inveniemus

$$\frac{dz}{dx}=\varphi(\nu)+x\varphi'(\nu)\frac{d\nu}{dx}+\psi'(\nu)\frac{d\nu}{dx},$$

$$\frac{dz}{dy}=x\varphi'(\nu)\frac{d\nu}{dy}+\psi'(\nu)\frac{d\nu}{dy}.$$

Istarum secunda multiplicata per $\frac{d\nu}{dx}$ subtrahatur ex prima ducta in $\frac{d\nu}{dy}$; prodibit

$$\frac{dz}{dx}\cdot\frac{d\nu}{dy}-\frac{dz}{dy}\cdot\frac{d\nu}{dx}=\varphi(\nu)\cdot\frac{d\nu}{dy}\ldots(m).$$

Rursus partialia aequationis (m) differentialia sumentes assequemur

$$\frac{d^2z}{dx^2}\cdot\frac{d\nu}{dy}+\frac{d^2\nu}{dxdy}\cdot\frac{dz}{dx}-\frac{d^2z}{dxdy}\cdot\frac{d\nu}{dx}-$$

$$\frac{d^2\nu}{dx^2}\cdot\frac{dz}{dy}=\varphi(\nu)\frac{d^2\nu}{dxdy}+\varphi'(\nu)\frac{d\nu}{dx}\cdot\frac{d\nu}{dy},$$

$$\frac{d^2v}{dy^2}\cdot\frac{dz}{dx}+\frac{d^2z}{dxdy}\cdot\frac{dv}{dy}-\frac{d^2v}{dxdy}\cdot\frac{dz}{dy}-$$

$$\frac{d^2z}{dy^2}\cdot\frac{dv}{dx}=\varphi(v)\frac{d^2v}{dy^2}+\varphi'(v)\left(\frac{dv}{dy}\right)^2:$$

exinde vero eliminata $\varphi'(v)$, exsurget aequatio solam continens indeterminatam $\varphi(v)$. Tum ab aequatione hoc pacto definita et ab (m) poterit expelli $\varphi(v)$, sicque ad partialem secundi ordinis aequationem differentialem perveniri arbitrariis functionibus liberatam.

55. Quoniam reiteratis differentiationibus novae in aequationes introducantur arbitrariae functiones φ'; φ'', ... ψ', ψ'', ... quisque videt, ad eliminandas functiones indeterminatas, plures requiri aequationes quam in eliminatione constantium quantitatum. Generatim si primitiva inter m variabiles x, y, z, ... aequatio complectitur p functiones arbitrarias expellendas φ, ψ, χ, ..., partiales primi ordinis aequationes praeter φ, ψ, χ, ... continebunt etiam φ', ψ', χ', ...; partiales secundi ordinis aequationes comprehendent amplius φ'', ψ'', χ'', ... atque ita porro, ut

$$p.(n+1)$$

exprimat numerum indeterminatarum functionum quae in primitivam aequationem simul et in derivatas partiales, ad ordinem usque n^{simum} perductas, ingrediuntur. Omnium autem ejusmodi aequationum numerus est (53)

$$\frac{(n+1)(n+2)(n+3)\dots(n+m-1)}{1.2.3\dots(m-1)};$$

poterit igitur ex iis aequationibus elici partialis, ordinis n^{simi}, differentialis aequatio functionibus arbitrariis exspoliata, quotiescumque fuerit

$$p(n+1) < \frac{(n+1)(n+2)(n+3)\ldots(n+m-1)}{1.2.3\ldots(m-1)},$$

seu

$$p < \frac{(n+2)(n+3)\ldots(n+m-1)}{1.2.3\ldots(m-1)}.$$

Ponantur v. gr. ternae dumtaxat variabiles x, y, z, ideoque $m=3$; prodibit

$$p < \frac{n+2}{2}, \text{ et consequenter } n > 2p-2.$$

Erunt nimirum protrahendae differentiationes in ordinem $2p-1$, ut arbitrariae functiones prorsus eliminentur.

Facto insuper $p=2$, tertius sese exhibet ordo. Equidem in aequatione (54 II.°)

$$z = x\varphi(v) + \psi(v)$$

habemus $p=2$, cum tamen satis fuerit differentiationes ad ordinem dumtaxat secundum perducere: id vero pendet a peculiari terminorum aequationis dispositione, ex qua fit in exemplo illo et in aliis similibus ut, certis adhibitis operationibus, plures simul dispareant arbitrariae functiones.

DE MAXIMIS MINIMISQUE VALORIBUS FUNCTIONUM QUAE EX PLURIBUS COALESCUNT VARIABILIBUS.

56. Circa functionem $\varphi(\theta)$ jam consideratam (47) liquet profecto illud: peculiari valori $\theta=0$ (in cujus viciniis ipsam $\varphi(\theta)$ ponimus esse continuam) nequit respondere maxima vel minima $\varphi(0)$, nisi vigeat aequatio (30)

$$\varphi'(0) = 0.$$

Ut autem $\varphi(0)$ sit revera maxima vel minima, necesse insuper est maneat constanter in iisdem viciniis

$$\text{vel } \varphi(\theta)-\varphi(0) < 0, \text{ vel } \varphi(\theta)-\varphi(0) > 0 :$$

in primo casu certe $\varphi(0)$ prodibit maxima, in secundo minima.

Quibus positis, confestim intelligitur (47. h''') functionem $\mu=f(x, y, z, \ldots)$ variabilium independentium $x, y, z, \ldots$ haud evasuram maximam minimamve nisi ob ejusmodi peculiares valores x_m, y_m, z_m, ..., quibus adhibitis loco x, y, z, ..., satisfiat aequationi

$$d\mu=0 ;$$

et quoniam huic satisfaciendum, utcumque caeteroquin se habent dx, dy, dz, ... quantitates invicem non dependentes, idcirco ex $d\mu=0$ profluent (45. h') aequationes

$$\frac{d\mu}{dx}=0, \quad \frac{d\mu}{dy}=0, \quad \frac{d\mu}{dz}=0, \text{ et caet.} \ldots \quad (g)$$

praebiturae valores x_m, y_m, z_m, ... qui functionem μ maximam minimamve poterunt constituere.

Ut vero μ prodeat reipsa maxima vel minima, oportet insuper in viciniis $\theta=0$ sit constanter vel

$$\left.\begin{array}{l} f(x_m+\theta dx, y_m+\theta dy, z_m+\theta dz, \ldots)- \\ \qquad f(x_m, y_m, z_m, \ldots) < 0, \\ \text{vel} \\ f(x_m+\theta dx, y_m+\theta dy, z_m+\theta dz, \ldots)- \\ \qquad f(x_m, y_m, z_m, \ldots) > 0, \end{array}\right\} (g')$$

quaecumque demum existant dx, dy, dz, ...; in primo casu haud dubie μ erit maxima, in secundo minima.

Exemplum.

Si detur functio $\mu = xy\,(3a - x - y)$, erunt

$$\frac{d\mu}{dx} = y(3a - 2x - y) = 0\,,$$

$$\frac{d\mu}{dy} = x(3a - 2y - x) = 0\,,$$

unde

$$x_m = a\,,\; y_m = a.$$

Differentia vero

$$(x_m + \theta dx)\,(y_m + \theta dy)\,(3a - x_m - \theta dx -$$
$$y_m - \theta dy) - x_m y_m\,(3a - x_m - y_m)\,,$$

seu

$$- a\theta^2\,[\,(dx + dy)^2 - dxdy\,] - \theta^3 dxdy\,(dx + dy)$$

permanet (utcumque se habent dx, dy) negativa in viciniis $\theta = 0$ si $a > 0$, positiva si $a < 0$; valores itaque $x_m = a$, $y_m = a$ praebent μ maximam in primo casu, minimam in secundo.

57. Si inter variabiles x, y, z, ... (quarum numerum ponimus $= k$) vigeant l relationes expressae per

$$v = 0\,,\; u = 0\,,\; s = 0\,, \ldots,$$

supererunt (1) $k - l$ variabiles independentes, totidemque proinde differentialia similiter independentia quibus applicanda tradita methodus.

Exemplum.

Proponatur functio

$$\mu = x^2 + y^2 + z^2 + \ldots,$$

atque inter variabiles x, y, z, ... vigeat relatio.

$$ax + by + cz + \ldots = h.$$

Erit

$$d\mu = 2xdx + 2ydy + 2zdz + \ldots,$$

itemque

$$adx + bdy + cdz + \ldots = 0,$$

$$dz = -\frac{adx + bdy + \ldots}{c}.$$

Quocirca

$$d\mu = 2\left(x - \frac{az}{c}\right)dx + 2\left(y - \frac{bz}{c}\right)dy + \ldots,$$

et consequenter

$$\frac{d\mu}{dx} = 2\left(x - \frac{az}{c}\right) = 0,\ \frac{d\mu}{dy} = 2\left(y - \frac{bz}{c}\right) = 0, \ldots.$$

unde

$$x = \frac{az}{c},\ y = \frac{bz}{c}, \ldots.$$

Non pluribus opus est ut intelligamus fore

$$z_m = \frac{ch}{a^2+b^2+c^2+\ldots},\ x_m = \frac{ah}{a^2+b^2+c^2+\ldots},$$

$$y_m = \frac{bh}{a^2+b^2+c^2+\ldots}, \ldots.$$

Jam vero differentia

$$\left.\begin{array}{r}(x_m + \theta dx)^2 + (y_m + \theta dy)^2 + \\ (z_m + \theta dz)^2 + \ldots - x^2_m - y^2_m - z^2_m - \ldots\end{array}\right\} =$$

$$\left.\begin{array}{l}\frac{2h\theta}{a^2+b^2+c^2+\dots}\Big(adx+bdy+\dots \\ -c\Big(\frac{adx+bdy+\dots}{c}\Big)\Big)+\theta^2(dx^2+ \\ dy^2+\dots)+\theta^2\Big(\frac{adx+bdy+\dots}{c}\Big)^2\end{array}\right\}=$$

$$\theta^2\left(dx^2+dy^2+\dots+\Big(\frac{adx+bdy+\dots}{c}\Big)^2\right)>0.$$

Valores igitur x_m, y_m, ... praebent minimam

$$\mu=x^2_m+y^2_m+z^2_m+\dots=\frac{h^2}{a^2+b^2+c^2+\dots}.$$

58. In definiendis maximis minimisve valoribus functionum plures complectentium variabiles possunt etiam adhiberi variorum ordinum differentialia.

Resumpta (56) functione $\varphi(\theta)$, respondebit valori $\theta=0$ maxima vel minima $\varphi(0)$, quotiescumque et $\varphi'(0)=0$, et ex caeteris derivatis (31)

$$\varphi''(0),\ \varphi'''(0),\ \varphi^{\mathrm{IV}}(0),\ \dots$$

quae non evanescit prima, est ordinis paris. Exprimat $\varphi^{(2n)}(0)$ derivatam illam; prodibit $\varphi(0)$ maxima si $\varphi^{(2n)}(0)<0$, minima si $\varphi^{(2n)}(0)>0$.

Hisce annotatis, quisque videt (47. h'''), functionem $f(x, y, z, \dots \tau)$ variabilium independentium $x, y, z, \dots \tau$ haud evasuram maximam minimamve nisi ob ejusmodi valores x_m, y_m, z_m, ... τ_m, ut et satisfiat aequationi $d\mu=0$ id est aequationibus (g. 56), et ex differentialibus

$$d^2\mu,\ d^3\mu,\ d^4\mu,\ \dots$$

quod non evanescit primum, sit ordinis paris utcumque alioquin se habent dx, dy, dz, ... $d\tau$. Exhibeatur differentiale illud per $d^{2n}\mu$; functio μ prodibit

$$\left.\begin{array}{l}\text{maxima si fuerit constanter } d^{2n}\mu < 0\,, \\ \text{minima si } \ldots\ldots\ldots\ldots\; d^{2n}\mu > 0\end{array}\right\}(g'').$$

Habemus autem (46)

$$d^{2n}\mu = \frac{d^{2n}\mu}{dx^{2n}}dx^{2n} + \frac{d^{2n}\mu}{dy^{2n}}dy^{2n} + \frac{d^{2n}\mu}{dz^{2n}}dz^{2n} + \ldots +$$

$$\frac{d^{2n}\mu}{d\tau^{2n}}d\tau^{2n} + 2n\frac{d^{2n}\mu}{dx\,dy^{2n-1}}dx\,dy^{2n-1} + \ldots\ldots$$

Designatis praeterea per p, q, r, ... rationibus

$$dx : d\tau,\ dy : d\tau,\ dz : d\tau,\ \ldots$$

et adhibita divisione per $d\tau^{2n}$, secundum membrum istius aequationis fiet

$$\left.\begin{array}{l}\dfrac{d^{2n}\mu}{dx^{2n}}p^{2n} + \dfrac{d^{2n}\mu}{dy^{2n}}q^{2n} + \dfrac{d^{2n}\mu}{dz^{2n}}r^{2n} + \ldots + \\ \quad\dfrac{d^{2n}\mu}{d\tau^{2n}} + 2n\dfrac{d^{2n}\mu}{dx\,dy^{2n-1}}pq^{2n-1} + \ldots\end{array}\right\}(K).$$

Itaque cum polynomium (K) eodem gaudeat signo atque $d^{2n}\mu$, in id recident conditiones (g''), ut permaneat (K) vel < 0, vel > 0, quaecumque sint p, q, r, ...

Caeterum constat (141 ex p. 1ª.) signum polynomii (K) haud mutatum iri, quotiescumque aequatio (K) $= 0$ resoluta v. gr. quoad p praebebit vel omnes radices imaginarias, vel, si quae sunt reales, eas et aequales, et numero pari, utcumque alioquin se habent q, r, ...

Hoc autem posito, signum polynomii (K) erit (131. 2°. ex p. 1ª.) constanter idem ac signum coefficientis $\dfrac{d^{2n}\mu}{dx^{2n}}$.

59. Sit v. gr.

$$\mu = f(x, y).$$

Differentiale secundi ordinis (46)

$$d^2\mu = \frac{d^2\mu}{dx^2}dx^2 + \frac{d^2\mu}{dy^2}dy^2 + 2\frac{d^2\mu}{dxdy}dxdy$$

eodem gaudebit signo ac trinomium

$$\frac{d^2\mu}{dx^2}p^2 + 2\frac{d^2\mu}{dxdy}p + \frac{d^2\mu}{dy^2} \dots (K):$$

insuper si binae radices aequationis

$$\frac{d^2\mu}{dx^2}p^2 + 2\frac{d^2\mu}{dxdy}p + \frac{d^2\mu}{dy^2} = 0$$

erunt vel imaginariae, vel reales simul et aequales, profecto variata p non idcirco mutabitur (141 ex p. 1.[a]) signum trinomii (K), ac proinde neque signum differentialis $d^2\mu$. Cum igitur illae binae radices prodeant imaginariae ubi (135 ex p. 1.[a])

$$\frac{d^2\mu}{dx^2} \cdot \frac{d^2\mu}{dy^2} - \left(\frac{d^2\mu}{dxdy}\right)^2 > 0,$$

prodeant vero reales simul et aequales ubi

$$\frac{d^2\mu}{dx^2} \cdot \frac{d^2\mu}{dy^2} - \left(\frac{d^2\mu}{dxdy}\right)^2 = 0,$$

sequitur sub altera ex hisce duabus conditionibus fore constanter $d^2\mu < 0$ si $\frac{d^2\mu}{dx^2} < 0$, fore autem constanter $d^2\mu > 0$ si $\frac{d^2\mu}{dx^2} > 0$, utcumque alioquin se habent dx, dy.

60. Proponatur quoque functio

$$\mu = f(x\,,\,y\,,\,z).$$

Differentiale secundi ordinis (46)

$$d^2\mu = \frac{d^2\mu}{dx^2}dx^2 + \frac{d^2\mu}{dy^2}dy^2 + \frac{d^2\mu}{dz^2}dz^2 +$$

$$2\frac{d^2\mu}{dxdy}dxdy + 2\frac{d^2\mu}{dxdz}dxdz + 2\frac{d^2\mu}{dydz}dydz$$

eodem afficietur signo atque polynomium

$$\frac{d^2\mu}{dx^2}p^2 + \frac{d^2\mu}{dy^2}q^2 + \frac{d^2\mu}{dz^2} + 2\frac{d^2\mu}{dxdy}pq +$$

$$2\frac{d^2\mu}{dxdz}p + 2\frac{d^2\mu}{dydz}q \ldots \text{(K)}.$$

Facto (K) $= 0$, et resoluta aequatione quoad p, prodibunt binae radices imaginariae si

$$\left.\begin{array}{l} \dfrac{d^2\mu}{dx^2}\cdot\dfrac{d^2\mu}{dz^2} - \left(\dfrac{d^2\mu}{dxdz}\right)^2 - \left(\left(\dfrac{d^2\mu}{dxdy}\right)^2 - \right. \\ \left.\dfrac{d^2\mu}{dx^2}\cdot\dfrac{d^2\mu}{dy^2}\right)q^2 - 2\left(\dfrac{d^2\mu}{dxdy}\cdot\dfrac{d^2\mu}{dxdz} - \dfrac{d^2\mu}{dx^2}\cdot\dfrac{d^2\mu}{dydz}\right)q \end{array}\right\} > 0;$$

prodibunt reales simul et aequales si haec ipsa quantitas $= 0$. Quibus conditionibus cum satisfieri debeat utcumque se habet q, propterea quoad primam erunt (131. 2° : 135 : 141 ex p. 1^a.)

$$\left(\frac{d^2\mu}{dxdy}\right)^2 - \frac{d^2\mu}{dx^2}\cdot\frac{d^2\mu}{dy^2} < 0\,,$$

$$\left.\begin{array}{l} \left(\left(\dfrac{d^2\mu}{dxdy}\right)^2 - \dfrac{d^2\mu}{dx^2}\cdot\dfrac{d^2\mu}{dy^2}\right)\left(\left(\dfrac{d^2\mu}{dxdz}\right)^2 - \right. \\ \left.\dfrac{d^2\mu}{dx^2}\cdot\dfrac{d^2\mu}{dz^2}\right) - \left(\dfrac{d^2\mu}{dxdy}\cdot\dfrac{d^2\mu}{dxdz} - \dfrac{d^2\mu}{dx^2}\cdot\dfrac{d^2\mu}{dydz}\right) \end{array}\right\} > 0;$$

quoad secundam vero

$$\left(\frac{d^2\mu}{dxdy}\right)^2 - \frac{d^2\mu}{dx^2}\cdot\frac{d^2\mu}{dy^2} = 0\,,\ \frac{d^2\mu}{dxdy}\cdot\frac{d^2\mu}{dxdz} -$$

$$\frac{d^2\mu}{dx^2}\cdot\frac{d^2\mu}{dydz} = 0\,,\ \frac{d^2\mu}{dx^2}\cdot\frac{d^2\mu}{dz^2} - \left(\frac{d^2\mu}{dxdz}\right)^2 = 0.$$

Adimpletis vel duabus illis, vel hisce tribus conditionibus, signum polynomii (K) permanebit idem, tametsi variantur p, q; ideoque perstabit etiam idem signum differentiali $d^2\mu$, quaecumque sint dx, dy, dz; eritque constanter aut $d^2\mu < 0$ si $\frac{d^2\mu}{dx^2} < 0$, aut $d^2\mu > 0$ si $\frac{d^2\mu}{dx^2} > 0$.

61. Ubi aliquae ex variabilibus x, y, z, ... desinant esse independentes, hic quoque servanda quae diximus (57); caeteris videlicet applicanda methodus.

Exemplum.

Sit functio

$$\mu = ax + by + cz\,,$$

vigeatque inter variabiles x, y, z relatio

$$x^2 + y^2 + z^2 = 1.$$

Differentialia capientes habebimus

$$d\mu = adx + bdy + cdz\,,\ xdx + ydy + zdz = 0\ ;\ \text{et}$$

$$dz = -\frac{xdx + ydy}{z}.$$

Hinc

$$d\mu = \left(a - \frac{cx}{z}\right)dx + \left(b - \frac{cy}{z}\right)dy\,;$$

$$\frac{d\mu}{dx} = a - \frac{cx}{z} = 0\,,\ \frac{d\mu}{dy} = b - \frac{cy}{z} = 0\,,$$

$$x = \frac{az}{c}\,,\ y = \frac{bz}{c}\,;$$

ideoque

$$z_m = \pm \frac{c}{\sqrt{a^2+b^2+c^2}}\,,\ x_m = \pm \frac{a}{\sqrt{a^2+b^2+c^2}}\,,$$

$$y_m = \pm \frac{b}{\sqrt{a^2+b^2+c^2}}\,.$$

Est autem $d^2\mu = cd^2z = -c\dfrac{dx^2+dy^2}{z} -$

$$c\frac{(xdx+ydy)^2}{z^3}\ldots(o)\,,$$

unde

$$\frac{d^2\mu}{dx^2} = -c\left(\frac{1}{z} + \frac{x^2}{z^3}\right)\,,\ \frac{d^2\mu}{dy^2} = -c\left(\frac{1}{z} + \frac{y^2}{z^3}\right)\,,$$

$$\frac{d^2\mu}{dxdy} = -c\frac{xy}{z^3}\,,$$

seu, adhibitis x_m, y_m, z_m pro x, y, z,

$$\frac{d^2\mu}{dx^2} = \mp\left(1 + \frac{a^2}{c^2}\right)\sqrt{a^2+b^2+c^2}\,,$$

$$\frac{d^2\mu}{dy^2} = \mp\left(1 + \frac{b^2}{c^2}\right)\sqrt{a^2+b^2+c^2}\,,$$

$$\frac{d^2\mu}{dxdy} = \mp\frac{ab}{c^2}\sqrt{a^2+b^2+c^2}\,;$$

et consequenter (59)

$$\left(\frac{d^2\mu}{dxdy}\right)^2 - \frac{d^2\mu}{dx^2}\cdot\frac{d^2\mu}{dy^2} = -\left(\frac{a^2+b^2+c^2}{c}\right)^2 < 0;$$

acceptis insuper signis superioribus in x_m, y_m, z_m, est $\frac{d^2\mu}{dx^2} < 0$; acceptis vero inferioribus, est $\frac{d^2\mu}{dx^2} > 0$: ergo (58) ex valoribus x_m, y_m, z_m positivis eruitur maxima $\mu = \sqrt{a^2+b^2+c^2}$, ex negativis minima $\mu = -\sqrt{a^2+b^2+c^2}$. Caeterum poterat id ipsum immediate deduci ex valore (o) differentialis $d^2\mu$; is enim, adhibitis x_m, y_m, z_m cum duplici signo $\pm$, vertitur in

$$d^2\mu = \mp (a^2+b^2+c^2)^{\frac{1}{2}}[dx^2+dy^2+(\frac{adx+bdy}{c})^2].$$

FORMULAE TAYLORI ET MAC-LAURINI EXTENDUNTUR AD FUNCTIONES PLURIUM VARIABILIUM: THEOREMA FUNCTIONUM HOMOGENEARUM DEMONSTRATUR.

62. Quoniam (23. q^{IX})

$$\varphi(\theta) = \varphi(o) + \frac{\theta}{1}\varphi'(o) + \frac{\theta^2}{1.2}\varphi''(o) + \frac{\theta^3}{1.2.3}\varphi'''(o) + \ldots$$

$$+ \frac{\theta^{m-1}}{1.2\ldots(m-1)}\varphi^{(m-1)}(o) + \frac{\theta^m}{1.2\ldots m}\varphi^{(m)}(\varepsilon\theta),$$

ideo (47. h''. h''')

$$f(x+\theta dx,\ y+\theta dy,\ z+\theta dz,\ \ldots) = \mu + \frac{\theta}{1}d\mu +$$

$$\frac{\theta^2}{1.2}d^2\mu + \frac{\theta^3}{1.2.3}d^3\mu + \ldots + \frac{\theta^{m-1}}{1.2\ldots(m-1)}d^{m-1}\mu +$$

$$\frac{\theta^m}{1.2\ldots m}\varphi^{(m)}(\varepsilon\theta)\ \ldots\ (o);$$

et consequenter si, aucto m indefinite, vergit

$$\frac{\theta^m}{1.2\ldots m}\varphi^{(m)}(\varepsilon\theta)$$

ad $lim. = 0$, scribi poterit aequatio

$$f(x+\theta dx\,,\,y+\theta dy\,,\,z+\theta dz\,,\ldots) = \mu + \frac{\theta}{1}d\mu + \frac{\theta^2}{1.2}d^2\mu + \frac{\theta^3}{1.2.3}d^3\mu + \ldots (o').$$

63. In (o') substitue valores $d\mu$, $d^2\mu$, ... (45:46); positis

$$\theta dx = \delta_1\,,\ \theta dy = \delta_2\,,\ \theta dz = \delta_3\,,\ \ldots$$

habebis

$$\left.\begin{aligned}
&f(x+\delta_1\,,\,y+\delta_2\,,\,z+\delta_3\,,\ldots) = \mu + \delta_1\frac{d\mu}{dx} + \\
&\delta_2\frac{d\mu}{dy} + \delta_3\frac{d\mu}{dz} + \ldots + \frac{\delta_1^{\,2}}{2}\frac{d^2\mu}{dx^2} + \frac{\delta^2_{\,2}}{2}\frac{d^2\mu}{dy^2} + \\
&\frac{\delta^2_{\,3}}{2}\frac{d^2\mu}{dz^2} + \ldots + \delta_1\delta_2\frac{d^2\mu}{dxdy} + \delta_1\delta_3\frac{d^2\mu}{dxdz} + \ldots \\
&+ \delta_2\delta_3\frac{d^2\mu}{dydz} + \ldots\ldots + \frac{\delta^3_{\,1}}{2.3}\frac{d^3\mu}{dx^3} + \ldots :
\end{aligned}\right\}(o'')$$

aequatio (o'') nihil est aliud nisi formula Taylori ad functiones plurium variabilium extensa.

Mutatis prius x, y, z, ... in zero, ac dein δ_1, δ_2, δ_3, ... in x, y, z, ..., proveniet formula *Mac-Laurini* ad easdem extensa functiones.

64. Functio $f(x\,,\,y\,,\,z\,,\ldots)$ dicitur *homogenea* quotiescumque, variatis x, y, z, ... in eadem ratione, ut evadant

$$vx\,,\ vy\,,\ vz\,,\ldots,$$

exsistet

$$f(vx\,,\ vy\,,\ vz\,,\ldots) = v^k f(x\,,\ y\,,\ z\,,\ldots) \ldots (\mathrm{o}''') :$$

exponens k denotat *gradum* functionis homogeneae.

Jam circa ejusmodi functiones facile demonstratur illud: homogeneitas functionis $f(x\,,y\,,\,z\,,\ldots)$ importat formulam (41)

$$\left.\begin{array}{l} xf'_x(x\,,y\,,\,z\,,\ldots) + yf'_y(x\,,\,y\,,\,z\,,\ldots) + \\ zf'_z(x\,,\,y\,,\,z\,,\ldots) + \ldots = kf(x\,,y\,,z\,,\ldots). \end{array}\right\} (\mathrm{o}^{\mathrm{IV}})$$

Differentiata enim (o''') quoad solam v, emerget (48)

$$f'_{vx}(vx\,,\ vy\,,\ vz\,,\ldots)xdv + f'_{vy}(vx\,;\ vy\,;\ vz\,,\ldots)ydv +$$

$$f'_{vz}(vx\,,\ vy\,,\ vz\,,\ldots)zdv + \ldots = kv^{k-1}f(x\,,\ y\,,\ z\,,\ldots)dv\,;$$

quae recidet in $(\mathrm{o}^{\mathrm{IV}})$, ubi deleatur prius communis factor dv, ac dein sumatur $v=1$.

Exsistente $k=0$, habebimus

$$\left.\begin{array}{l} xf'_x(x\,;y\,,z\,,\ldots) + yf'_y(x\,,y\,,\,z\,,\ldots) + \\ zf'_z(x\,,\,y\,,\ z\,,\ldots) + \ldots = 0 \end{array}\right\} (\mathrm{o}^{\mathrm{V}})\,;$$

uti videre est in functione $\mathrm{L}\left(\dfrac{x}{y}\right)$.

DE FUNCTIONUM RESIDUIS: UBI ET DE FRACTIONUM RATIONALIUM DECOMPOSITIONE IN ALIAS SIMPLICIORES, ET DE SERIE LAGRANGIANA.

65. Denotent x_1, x_2, x_3, ... radices aequationis

$$\frac{1}{f(x)}=0 \ldots (g);$$

ac 1°. ponantur inaequales. Etsi

$f(x_1)=\infty$, $f(x_2)=\infty$, $f(x_3)=\infty$, et caet. ...;

poterunt tamen (27)

$(x_1-x_1)f(x_1)$, $(x_2-x_2)f(x_2)$, $(x_3-x_3)f(x_3)$, et caet...

determinatum obtinere valorem. Fiant in hac hypothesi

$$\left.\begin{array}{l}(x-x_1)f(x)=\mathrm{f}(x),\ (x-x_2)f(x)=\mathrm{f}_1(x);\ \text{et caet...},\\ \text{unde}\\ f(x)=\dfrac{\mathrm{f}(x)}{x-x_1},\ f(x)=\dfrac{\mathrm{f}_1(x)}{x-x_2};\ \text{et caet...}\end{array}\right\}(g')$$

et designante θ infinitesimam quantitatem, substitue $x_1+\theta$ loco x in prima, $x_2+\theta$ in secunda, et caet...; habebis (23. q^{VIII})

$$\left.\begin{array}{l}f(x_1+\theta)=\dfrac{\mathrm{f}(x_1+\theta)}{\theta}=\dfrac{1}{\theta}\mathrm{f}(x_1)+\mathrm{f}'(x_1+\varepsilon\theta),\\ f(x_2+\theta)=\dfrac{\mathrm{f}_1(x_2+\theta)}{\theta}=\dfrac{1}{\theta}\mathrm{f}_1(x_2)+\mathrm{f}'_1(x_2+\varepsilon\theta),\end{array}\right\}(g'')$$

et caet. ...

2°. aequationi (g) sint aut duae, aut tres, aut plures, ac generatim n radices $=x_1$, m radices $=x_2$, et caet. ...; positis

$$\left.\begin{array}{l}(x-x_1)^n f(x)=\mathrm{f}(x)\,,\ (x-x_2)^m f(x)=\mathrm{f}_1(x)\,,\\ \qquad\text{et caet. ...., unde}\\ f(x)=\dfrac{\mathrm{f}(x)}{(x-x_1)^n}\,,\ f(x)=\dfrac{\mathrm{f}_1(x)}{(x-x_2)^m},\ \text{et caet. ...,}\end{array}\right\}(g^{\text{III}})$$

et adhibitis respective $x_1+\theta$, $x_2+\theta$, ... pro x, prodibit (23. q^{VIII})

$$\left.\begin{array}{l}f(x_1+\theta)=\dfrac{\mathrm{f}(x_1+\theta)}{\theta^n}=\dfrac{1}{\theta^n}\mathrm{f}(x_1)+\dfrac{1}{\theta^{n-1}}\dfrac{\mathrm{f}'(x_1)}{1}+\\ \dfrac{1}{\theta^{n-2}}\dfrac{\mathrm{f}''(x_1)}{1.2}+\ldots+\dfrac{1}{\theta}\dfrac{\mathrm{f}^{(n-1)}(x_1)}{1.2\ldots(n-1)}+\dfrac{\mathrm{f}^{(n)}(x_1+\varepsilon\theta)}{1.2\ldots n},\end{array}\right\}(g^{\text{IV}})$$

et caet. ....

Coefficientes

$$\mathrm{f}(x_1)\,,\ \mathrm{f}(x_2)\,,\ \ldots.$$

quantitatis $\dfrac{1}{\theta}$ in primo casu, coefficientes vero

$$\frac{\mathrm{f}^{(n-1)}(x_1)}{1.2\ldots(n-1)}\,,\ \frac{\mathrm{f}^{(m-1)}(x_2)}{1.2\ldots(m-1)}\,,\ \ldots$$

ipsius $\dfrac{1}{\theta}$ in secundo vocantur a D. Cauchy *residua* functionis $f(x)$ quoad $x=x_1$, $x=x_2$; ...

66. Determinantes istiusmodi coefficientes dicimur *extrahere* residua e functione $f(x)$ seu, quod eodem redit, ex

$$\frac{(x-x_1)f(x)}{x-x_1}\,,\ \frac{(x-x_2)f(x)}{x-x_2}\,,\ \ldots;$$

residuorum insuper extractio indicatur littera $\mathcal{E}$ ita,

ut expressiones

$$\mathcal{E}\frac{(x-x_1)f(x)}{((x-x_1))}\,, \quad \mathcal{E}\frac{(x-x_2)\,f(x)}{((x-x_2))}\,, \ldots$$

significent *partialia* functionis $f(x)$ residua quoad $x=x_1$, quoad $x=x_2$, ... ; summa demum omnium residuorum quoad $x=x_1$, $x=x_2$, $x=x_3$, et caet. ... designatur per

$$\mathcal{E}((f(x))).$$

Quare ubi (g) careat aequalibus radicibus, erunt

$$\left.\begin{array}{l} \mathcal{E}\frac{(x-x_1)f(x)}{((x-x_1))}=\mathrm{f}(x_1),\ \mathcal{E}\frac{(x-x_2)f(x)}{((x-x_2))}=\mathrm{f}_1(x_2),\ \ldots, \\ \mathcal{E}((f(x)))=\mathrm{f}(x_1)+\mathrm{f}_1(x_2)+\mathrm{f}_2(x_3)+\ldots\ldots; \end{array}\right\}(g^{\mathrm{v}})$$

et positis n radicibus $=x_1$,

$$\left.\begin{array}{l} \mathcal{E}\frac{(x-x_1)f(x)}{((x-x_1))}=\mathcal{E}\frac{(x-x_1)^n f(x)}{(([x-x_1]^n))}=\frac{f^{(n-1)}(x_1)}{1.2\ldots(n-1)}\,, \\ \mathcal{E}((f(x)))=\frac{f^{(n-1)}(x_1)}{1.2\ldots(n-1)}+\mathrm{f}_1(x_2)+\mathrm{f}_2(x_3)+\ldots \end{array}\right\}(g^{\mathrm{vi}})$$

67. Haec notentur: 1.° ex (g'') habemus

$$\mathrm{f}(x_1)=lim.\theta[f(x_1+\theta)-\mathrm{f}'(x_1+\varepsilon\theta)]=lim.\theta f(x_1+\theta)\,,$$

seu

$$\mathcal{E}\frac{(x-x_1)f(x)}{((x-x_1))}=lim.\theta f(x_1+\theta).$$

2° hinc ubi peculiaris valor x_0 non det $f(x)=\infty$, erit

$$\mathcal{E}\frac{(x-x_0)f(x)}{((x-x_0))}=lim.\theta f(x_0+\theta)=0\,;$$

residuum videlicet functionis $f(x)$ quoad $x=x_1$ proveniet $=0$;

3.° si ponitur $f(x)=\frac{\chi(x)}{\varphi(x)}$, ac praeterea x_1 una e radicibus aequationis $\varphi(x)=0$; erit (1.°)

$$\mathrm{f}(x_1)=\lim.\theta\frac{\chi(x_1+\theta)}{\varphi(x_1+\theta)};$$

proinde (23. q^{VIII})

$$\mathrm{f}(x_1)=\lim.\theta\frac{\chi(x_1+\theta)}{\varphi(x_1)+\theta\varphi'(x_1+\varepsilon\theta)}=\frac{\chi(x_1)}{\varphi'(x_1)},$$

seu

$$\mathcal{E}\frac{(x-x_1)f(x)}{((x-x_1))}=\frac{\chi(x_1)}{\varphi'(x_1)}.$$

4°. quoniam (65. g^{IV})

$$\mathrm{f}(x_1+\theta)=\theta^n f(x_1+\theta),$$

ideo

$$\frac{\mathrm{f}^{(n-1)}(x_1)}{1.2\ldots(n-1)}=\frac{1}{1.2\ldots(n-1)}\cdot\frac{d^{n-1}[\theta^n f(x_1+\theta)]}{d\theta^{n-1}};$$

modo, absolutis differentiationibus, fiat $\theta=0$.

5° prima (g^{V}) et prima (g^{VI}) recidunt (65. g'. g''') in

$$\mathcal{E}\frac{\mathrm{f}(x)}{((x-x_1))}=\mathrm{f}(x_1),\quad \mathcal{E}\frac{\mathrm{f}(x)}{(([x-x_1]^n))}=\frac{\mathrm{f}^{(n-1)}(x_1)}{1.2\ldots(n-1)};$$

quae, adhibitis z et $n+1$ pro x et n, vertentur in

$$\mathcal{E}\frac{\mathrm{f}(z)}{((z-x_1))}=\mathrm{f}(x_1),\quad \mathcal{E}\frac{\mathrm{f}(z)}{(([z-x_1]^{n+1}))}=\frac{\mathrm{f}^{(n)}(x_1)}{1.2\ldots n}.$$

6.° quoniam (10)

$$\frac{d\left[\frac{\chi(z)}{z-x}\right]}{dz}=\frac{\chi'(z)}{z-x}-\frac{\chi(z)}{(z-x)^2}\ ,$$

$$\frac{d\left[\frac{\chi(z)}{(z-x)^n}\right]}{dz}=\frac{\chi'(z)}{(z-x)^n}-\frac{n\chi(z)}{(z-x)^{n-1}}\ ,$$

ideo (6.°) residua functionum

$$\frac{d\left[\frac{\chi(z)}{z-x}\right]}{dz}\ ,\quad \frac{d\left[\frac{\chi(z)}{(z-x)^n}\right]}{dz}$$

quoad $z=x$ erunt

$$\chi'(x)-\chi'(x)=0,\ \frac{\chi^{(n)}(x)}{1.2...(n-1)}-\frac{\chi^{(n)}(x)}{1.2...(n-1)}=0.$$

7.° quia igitur (9)

$$f(z)\frac{d\left[\frac{\chi(z)}{z-x}\right]}{dz}=\frac{d\left[\frac{f(z)\chi(z)}{z-x}\right]}{dz}-f'(z)\frac{\chi(z)}{z-x}\ ,$$

$$f(z)\frac{d\left[\frac{\chi(z)}{(z-x)^n}\right]}{dz}=\frac{d\left[\frac{f(z)\chi(z)}{(z-x)^n}\right]}{dz}-f'(z)\frac{\chi(z)}{(z-x)^n}\ ,$$

iccirco residua functionum

$$f(z)\frac{d\left[\frac{\chi(z)}{z-x}\right]}{dz}\ ,\quad f(z)\frac{d\left[\frac{\chi(z)}{(z-x)^n}\right]}{dz}$$

quoad $z=x$ erunt (6.°)

$$-\mathcal{E}\frac{f'(z)\chi(z)}{((z-x))}\text{ seu }(5.°)-f'(x)\chi(x),$$

$$-\mathcal{E}\frac{f'(z)\chi(z)}{(([z-x]^n))}\text{ seu }(5.°)-\frac{1}{1.2...(n-1)}\frac{d^{n-1}[f'(x)\chi(x)]}{dx^{n-1}}.$$

8.° posita $f(z)=\varphi(z)+\chi(z)+\dots$, erit (65 : 66)

$$\mathcal{E}((f(z)))=\mathcal{E}((\varphi(z)))+\mathcal{E}((\chi(z)))+\dots$$

68. In (g^{IV}) substitue $x-x_1$ loco θ; habebis

$$f(x)=\frac{\mathrm{f}(x_1)}{(x-x_1)^n}+$$

$$\frac{1}{1}\cdot\frac{\mathrm{f}'(x_1)}{(x-x_1)^{n-1}}+\frac{1}{1.2}\cdot\frac{\mathrm{f}''(x_1)}{(x-x_1)^{n-2}}+\dots.$$

$$+\frac{1}{1.2.3\dots(n-1)}\cdot\frac{\mathrm{f}^{(n-1)}(x_1)}{x-x_1}+$$

$$\frac{1}{1.2.3\dots n}\cdot\mathrm{f}^{(n)}[x_1+\varepsilon(x-x_1)].$$

Est autem (67. 5.° 8.°)

$$\frac{\mathrm{f}(x_1)}{(x-x_1)^n}+\frac{1}{1}\cdot\frac{\mathrm{f}'(x_1)}{(x-x_1)^{n-1}}+$$

$$\frac{1}{1.2}\cdot\frac{\mathrm{f}''(x_1)}{(x-x_1)^{n-2}}+\dots+\frac{1}{1.2.3\dots(n-1)}\cdot\frac{\mathrm{f}^{(n-1)}(x_1)}{x-x_1}=$$

$$\frac{1}{(x-x_1)^n}\mathcal{E}\frac{\mathrm{f}(z)}{((z-x_1))}+$$

$$\frac{1}{(x-x_1)^{n-1}}\mathcal{E}\frac{\mathrm{f}(z)}{(([z-x_1]^2))}+\dots+$$

$$\frac{1}{x-x_1}\mathcal{E}\frac{\mathrm{f}(z)}{(([z-x_1]^n))}=$$

$$\mathcal{E}\frac{f(z)}{(x-x_1)^n\,((\,[z-x_1]^n\,))}[(z-x_1)^{n-1}+$$

$$(x-x_1)(z-x_1)^{n-2}+(x-x_1)^2(z-x_1)^{n-3}+\ldots+$$

$$(x-x_1)^{n-1}]=$$

$$\mathcal{E}\frac{f(z)}{(x-x_1)^n((\,[z-x_1]^n\,))}\cdot\frac{(x-x_1)^n-(z-x_1)^n}{x-z}=$$

$$\mathcal{E}\frac{f(z)}{(x-z)((\,[z-x_1]^n))}-\frac{1}{(x-x_1)^n}\mathcal{E}\frac{(z-x_1)^n f(z)}{(x-z)((\,[z-x_1]^n))},$$

ubi

$$\mathcal{E}\frac{(z-x_1)^n f(z)}{(x-z)((\,[z-x_1]^n\,))},\ \text{seu}\ \mathcal{E}\frac{(z-x_1)f(z)}{(x-z)((z-x_1))}$$

exhibet residuum functionis

$$\frac{f(z)}{x-z}$$

relate ad $z=x_1$, quod residuum (67. 2.°) $=0$: igitur

$$\frac{f(x_1)}{(x-x_1)^n}+\frac{1}{1}\cdot\frac{f'(x_1)}{(x-x_1)^{n-1}}+$$

$$\frac{1}{1.2}\cdot\frac{f''(x_1)}{(x-x_1)^{n-2}}+\ldots+\frac{1}{1.2.3\ldots(n-1)}\cdot\frac{f^{(n-1)}(x_1)}{x-x_1}=$$

$$\mathcal{E}\frac{f(z)}{(x-z)((\,[z-x_1]^n))}.$$

Fiat

$$\frac{f^{(n)}(x_1+\varepsilon(x-x_1))}{1.2.3\ldots n}=\psi(x)\ldots(g^{\text{VII}});$$

proveniet

$$f(x) = \mathcal{E}\frac{f(z)}{(x-z)\,(([z-x_1]^n))} + \psi(x)\,;$$

ideoque (65. g''')

$$f(x) - \mathcal{E}\frac{(z-x_1)\,f(z)}{(x-z)\,((z-x_1))} = \psi(x) \ldots (g^{\mathrm{VIII}}).$$

69. Haec nunc facile stabiliuntur. 1.° sumpta $x = x_1$, erit (68. g^{VII})

$$\psi(x_1) = \frac{f^{(n)}(x_1)}{1.2.3\ldots n}\,;$$

functio videlicet $\psi(x_1)$ valorem obtinet finitum: proinde (68. g^{VIII}) etsi $f(x)$ evadit infinita quando $x = x_1$, tamen differentia inter $f(x_1)$ et residuum functionis

$$\frac{f(z)}{x-z}$$

relate ad $z = x_1$ manet quantitas finita sub eadem positione $x = x_1$. 2.° denotante igitur

$$\mathcal{E}\frac{((f(z)))}{x-z}$$

summam residuorum functionis

$$\frac{f(z)}{x-z}$$

quoad $z = x_1$, $z = x_2$, $z = x_3$, ... quantitas

$$f(x) - \mathcal{E}\frac{((f(z)))}{x-z}$$

nunquam evadet infinita, quaecumque caeteroquin e

radicibus x_1, x_2, x_3, ... aequationis (g) adhibeatur pro x. Quare si fiat

$$f(x) - \mathcal{E}\frac{((f(z)))}{x-z} = F(x) \ldots (g^{IX}),$$

functio $F(x)$ permanebit finita quoad $x=x_1$, $x=x_2$, $x=x_3$, ..., et consequenter (67. 2°.) quoad omnes quantitatis variabilis x valores finitos. 3.° si $f(x)$ est fractio rationalis, erit $F(x)$ ejusmodi fractio similiter rationalis in qua denominator nunquam fiet $=0$; habebit videlicet functio $F(x)$ denominatorem constantem, eritque proinde $F(x)$ functio integra quantitatis variabilis x. 4°. quibus positis, assumatur

$$f(x) = \frac{\chi(x)}{\varphi(x)},$$

ubi $\chi(x)$, $\varphi(x)$ designant functiones integras variabilis x, ponaturque gradus denominatoris $\varphi(x)$ major quam gradus numeratoris $\chi(x)$: existet

$$f(x) = 0, \text{ si } x = \infty,$$

totumque (67. 2.°) primum membrum aequationis (g^{IX}) evanescet, ideoque et secundum: quod cum sit functio integra quantitatis x, oportet in ea qua sumus hypothesi existat constanter $F(x)=0$, et consequenter

$$f(x) = \mathcal{E}\frac{((f(z)))}{x-z} \ldots (g^{X}).$$

5.° multiplicetur (g^{X}) per x, ut prodeat

$$x f(x) = \mathcal{E}\frac{((f(z)))}{1-\frac{z}{x}}:$$

facta $x=\infty$, exhibeatur valor $x f(x)$ per Θ; erit

$$\left.\begin{array}{r}\Theta = \mathcal{E}\,((f(z)))\,; \\ \text{et evanescente } \Theta\,, \\ \mathcal{E}\,((f(z))) = 0\,; \end{array}\right\}(g^{\text{xi}})$$

quarum secunda exposcit ut in fractione rationali

$$f(x) = \frac{\chi(x)}{\varphi(x)}$$

non modo gradus denominatoris superet numeratoris gradum, sed etiam differentia inter utrumque gradum sit > 1.

70. Data functione $f(x, y)$ binarum varibilium independentium x et y, ponatur aequatio

$$\frac{1}{f(x, y)} = 0$$

resoluta quoad x suppeditare radices $x_1, x_2, \ldots$; ac 1.° sit (65. g'')

$$f(x_1 + \theta, y + \theta') = \frac{\mathrm{f}(x_1 + \theta, y + \theta')}{\theta}\;;$$

erit (63. o'': 66. g^{v})

$$\mathcal{E}\,\frac{(x - x_1)\,f(x, y)}{((x - x_1))} = \mathrm{f}(x_1, y),$$

unde

$$\frac{d_y\,\mathcal{E}\,\dfrac{(x - x_1)\,f(x, y)}{((x - x_1))}}{dy} = \frac{d_y\,\mathrm{f}(x_1, y)}{dy}\;;$$

item (65. g')

$$\frac{d_y f(x, y)}{dy} = \frac{\dfrac{d_y \mathrm{f}(x, y)}{dy}}{x - x_1},$$

et consequenter (67. 4.°)

$$\mathcal{E}\,\frac{(x-x_1)\dfrac{d_y f(x\,,\,y)}{dy}}{((x-x_1))}=\frac{d_y \mathrm{f}(x\,,\,y)}{dy}\,,$$

2.° sit (65. g^{IV})

$$f(x_1+\theta\,,\,y+\theta')=\frac{\mathrm{f}(x_1+\theta\,,\,y+\theta')}{\theta^n}\,:$$

erit (63. o'' : 66. g^{VI})

$$\mathcal{E}\,\frac{(x-x_1)f(x\,,\,y)}{((x-x_1))}=\frac{1}{1.2\ldots(n-1)}\cdot\frac{d^{n-1}\mathrm{f}(x_1\,,\,y)}{dx_1^{\,n-1}}\,,$$

ac proinde

$$\frac{dy\,\mathcal{E}\,\dfrac{(x-x_1)f(x\,,\,y)}{((x-x_1))}}{dy}=\frac{d^n\mathrm{f}(x_1\,,\,y)}{dx_1^{\,n-1}dy}\,;$$

item (65. g''')

$$\frac{d_y f(x\,,\,y)}{dy}=\frac{\dfrac{d_y \mathrm{f}(x\,,\,y)}{dy}}{(x-x_1)^n}\,,$$

ideoque (67. 5.°)

$$\mathcal{E}\,\frac{(x-x_1)\dfrac{d_y f(x\,,\,y)}{dy}}{((x-x_1))}=\frac{d^n\mathrm{f}(x_1\,,\,y)}{dx_1^{\,n-1}dy}\,.$$

Non pluribus opus est ut intelligamus fore

$$\frac{d_y\,\mathcal{E}\,\dfrac{(x-x_1)f(x\,,\,y)}{((x-x_1))}}{dy}=\mathcal{E}\,\frac{(x-x_1)\dfrac{d_y f(x\,,\,y)}{dy}}{((x-x_1))}\,:$$

simili modo

$$\frac{d_y \mathcal{E}\,\dfrac{(x-x_2)f(x,y)}{((x-x_2))}}{dy}=\mathcal{E}\,\frac{(x-x_2)\dfrac{d_y f(x,y)}{dy}}{((x-x_2))},$$

atque ita porro. Hinc

$$\frac{d_y\mathcal{E}((f(x,y)))}{dy}=\mathcal{E}\left(\left(\frac{d_y f(x,y)}{dy}\right)\right)\ldots(g^{\text{XII}}).$$

71. Ope formulae (g^{X}) resolvuntur fractiones rationales in alias simpliciores: rem bina declarabunt exempla.

1.° Detur fractio

$$\frac{x^3+x^2+2}{x(x-1)^2(x+1)^2}:$$

erit

$$\frac{x^3+x^2+2}{x(x-1)^2(x+1)^2}=\mathcal{E}\left(\left(\frac{z^3+z^2+2}{z(z-1)^2(z+1)^2}\right)\right)\frac{1}{x-z}=$$

$$\mathcal{E}\frac{z^3+z^2+2}{((z))(z-1)^2(z+1)^2(x-z)}+\mathcal{E}\frac{z^3+z^2+2}{z(([z-1]^2))(z+1)^2(x-z)}+$$

$$\mathcal{E}\frac{z^3+z^2+2}{z(z-1)^2(([z+1]^2))(x-z)}.$$

Jam vero relate ad $z=x_1=0$ est (66. g^{V})

$$\mathcal{E}\frac{z^3+z^2+2}{((z))(z-1)^2(z+1)^2(x-z)}=f(x_1)=\frac{2}{x};$$

insuper quoad $z=x_2=1$ habemus (66. g^{VI}: 67. 4.°)

$$\mathcal{E}\frac{z^3+z^2+2}{((z))(z-1)^2\ (z+1)^2(x-z)}=f(x_1)=\frac{2}{x};$$

insuper quoad $z=x_2=1$ habemus (66. g^{vi}; 67. 4.°)

$$\mathcal{E}\,\frac{z^3+z^2+2}{z(([z-1]^2))(z+1)^2(x-z)}=\mathrm{f}'(x_1)=$$

$$\frac{d[\theta^2 f(x_2+\theta)]}{d\theta}=\frac{d\,\dfrac{(1+\theta)^3+(1+\theta)^2+2}{(1+\theta)(2+\theta)^2(x-1-\theta)}}{d\theta}=$$

$$\frac{1}{(x-1)^2}-\frac{3}{4(x-1)}\,;$$

denique quoad $z=x_3=-1$,

$$\mathcal{E}\,\frac{z^3+z^2+2}{z(z-1)^2(([z+1]^2))(x-z)}=$$

$$\frac{d\,\dfrac{(\theta-1)^3+(\theta-1)^2+2}{(\theta-1)(\theta-2)^2(x+1-\theta)}}{d\theta}=-\frac{1}{2(x+1)^2}-\frac{5}{4(x+1)}\,;$$

igitur

$$\frac{x^3+x^2+2}{x(x-1)^2(x+1)^2}=\frac{2}{x}+\frac{1}{(x-1)^2}-\frac{3}{4(x-1)}-$$

$$\frac{1}{2(x+1)^2}-\frac{5}{4(x+1)}.$$

II.° Detur fractio

$$\frac{1}{x^3(x^2+1)}\,;$$

erit

$$\frac{1}{x^3(x^2+1)}=\mathcal{E}\,\frac{1}{((z^3))(z^2+1)(x-z)}+$$

$$\left.\begin{array}{l}(x-x_1)^n f(x) = \mathrm{f}(x)\ ,\ (x-x_2)^m f(x) = \mathrm{f}_1(x)\ ,\\ \quad \text{et caet.}\ \ldots\ ,\ \text{unde}\\ f(x) = \dfrac{\mathrm{f}(x)}{(x-x_1)^n}\ ,\ f(x) = \dfrac{\mathrm{f}_1(x)}{(x-x_2)^m}\ ,\ \text{et caet.}\ \ldots\ ,\end{array}\right\}(g^{m})$$

et adhibitis respective $x_1+\theta$, $x_2+\theta$, ... pro x, prodibit (23. q^{VIII})

$$\left.\begin{array}{l} f(x_1+\theta) = \dfrac{\mathrm{f}(x_1+\theta)}{\theta^n} = \dfrac{1}{\theta^n}\mathrm{f}(x_1) + \dfrac{1}{\theta^{n-1}}\dfrac{\mathrm{f}'(x_1)}{1} + \\ \dfrac{1}{\theta^{n-2}}\dfrac{\mathrm{f}''(x_1)}{1.2} + \ldots + \dfrac{1}{\theta}\dfrac{\mathrm{f}^{(n-1)}(x_1)}{1.2\ldots(n-1)} + \dfrac{\mathrm{f}^{(n)}(x_1+\varepsilon\theta)}{1.2\ldots n}\ ,\end{array}\right\}(g^{\text{IV}})$$

et caet. ....

Coefficientes

$$\mathrm{f}(x_1)\ ,\ \mathrm{f}(x_2)\ ,\ \ldots$$

quantitatis $\dfrac{1}{\theta}$ in primo casu, coefficientes vero

$$\frac{\mathrm{f}^{(n-1)}(x_1)}{1.2\ldots(n-1)}\ ,\ \frac{\mathrm{f}^{(m-1)}(x_2)}{1.2\ldots(m-1)}\ ,\ \ldots$$

ipsius $\dfrac{1}{\theta}$ in secundo vocantur a D. Cauchy *residua* functionis $f(x)$ quoad $x=x_1$, $x=x_2$; ...

66. Determinantes istiusmodi coefficientes dicimur *extrahere* residua e functione $f(x)$ seu, quod eodem redit, ex

$$\frac{(x-x_1)f(x)}{x-x_1}\ ,\ \frac{(x-x_2)f(x)}{x-x_2}\ ,\ \ldots\ ;$$

residuorum insuper extractio indicatur littera $\mathcal{E}$ ita,

ut expressiones

$$\mathcal{E}\frac{(x-x_1)f(x)}{((x-x_1))}\,, \quad \mathcal{E}\frac{(x-x_2)\,f(x)}{((x-x_2))}\,, \ldots$$

significent *partialia* functionis $f(x)$ residua quoad $x = x_1$, quoad $x = x_2$, ... ; summa demum omnium residuorum quoad $x = x_1$, $x = x_2$, $x = x_3$, et caet. ... designatur per

$$\mathcal{E}((f(x))).$$

Quare ubi (g) careat aequalibus radicibus, erunt

$$\left.\begin{array}{l} \mathcal{E}\dfrac{(x-x_1)f(x)}{((x-x_1))} = \mathrm{f}(x_1),\ \mathcal{E}\dfrac{(x-x_2)f(x)}{((x-x_2))} = \mathrm{f}_1(x_2),\ \ldots, \\ \mathcal{E}((f(x))) = \mathrm{f}(x_1) + \mathrm{f}_1(x_2) + \mathrm{f}_2(x_3) + \ldots ; \end{array}\right\} \quad (g^{\mathrm{V}})$$

et positis n radicibus $= x_1$,

$$\left.\begin{array}{l} \mathcal{E}\dfrac{(x-x_1)f(x)}{((x-x_1))} = \mathcal{E}\dfrac{(x-x_1)^n f(x)}{(([x-x_1]^n))} = \dfrac{f^{(n-1)}(x_1)}{1.2\ldots(n-1)}\,, \\ \mathcal{E}((f(x))) = \dfrac{f^{(n-1)}(x_1)}{1.2\ldots(n-1)} + \mathrm{f}_1(x_2) + \mathrm{f}_2(x_3) + \ldots \end{array}\right\} \quad (g^{\mathrm{VI}})$$

67. Haec notentur : 1.° ex (g') habemus

$$\mathrm{f}(x_1) = lim.\theta[f(x_1+\theta) - \mathrm{f}'(x_1+\varepsilon\theta)] = lim.\theta f(x_1+\theta)\,,$$

seu

$$\mathcal{E}\frac{(x-x_1)f(x)}{((x-x_1))} = lim.\theta f(x_1+\theta).$$

2° hinc ubi peculiaris valor x_0 non det $f(x) = \infty$, erit

$$\mathcal{E}\frac{(x-x_0)f(x)}{((x-x_0))} = lim.\theta f(x_0+\theta) = 0\,;$$

residuum videlicet functionis $f(x)$ quoad $x=x_1$ proveniet $=0$;

3.° si ponitur $f(x)=\frac{\chi(x)}{\varphi(x)}$, ac praeterea x_1 una e radicibus aequationis $\varphi(x)=0$; erit (1.°)

$$f(x_1)=lim.\theta\frac{\chi(x_1+\theta)}{\varphi(x_1+\theta)};$$

proinde (23. q^{viii})

$$f(x_1)=lim.\theta\frac{\chi(x_1+\theta)}{\varphi(x_1)+\theta\varphi'(x_1+\varepsilon\theta)}=\frac{\chi(x_1)}{\varphi'(x_1)},$$

seu

$$\mathcal{E}\frac{(x-x_1)f(x)}{((x-x_1))}=\frac{\chi(x_1)}{\varphi'(x_1)}.$$

4°. quoniam (65. g^{iv})

$$f(x_1+\theta)=\theta^n f(x_1+\theta),$$

ideo

$$\frac{f^{(n-1)}(x_1)}{1.2\ldots(n-1)}=\frac{1}{1.2\ldots(n-1)}\cdot\frac{d^{n-1}[\theta^n f(x_1+\theta)]}{d\theta^{n-1}};$$

modo, absolutis differentiationibus, fiat $\theta=0$.

5° prima (g^{v}) et prima (g^{vi}) recidunt (65. $g'.g'''$) in

$$\mathcal{E}\frac{f(x)}{((x-x_1))}=f(x_1),\ \mathcal{E}\frac{f(x)}{(([x-x_1]^n))}=\frac{f^{(n-1)}(x_1)}{1.2\ldots(n-1)};$$

quae, adhibitis z et $n+1$ pro x et n, vertentur in

$$\mathcal{E}\frac{f(z)}{((z-x_1))}=f(x_1),\ \mathcal{E}\frac{f(z)}{(([z-x_1]^{n+1}))}=\frac{f^{(n)}(x_1)}{1.2\ldots n}.$$

6.° quoniam (10)

$$\frac{d\left[\frac{\chi(z)}{z-x}\right]}{dz}=\frac{\chi'(z)}{z-x}-\frac{\chi(z)}{(z-x)^2},$$

$$\frac{d\left[\frac{\chi(z)}{(z-x)^n}\right]}{dz}=\frac{\chi'(z)}{(z-x)^n}-\frac{n\chi(z)}{(z-x)^{n-1}},$$

ideo (5.°) residua functionum

$$\frac{d\left[\frac{\chi(z)}{z-x}\right]}{dz},\quad \frac{d\left[\frac{\chi(z)}{(z-x)^n}\right]}{dz}$$

quoad $z=x$ erunt

$$\chi'(x)-\chi'(x)=0,\quad \frac{\chi^{(n)}(x)}{1.2\ldots(n-1)}-\frac{\chi^{(n)}(x)}{1.2\ldots(n-1)}=0.$$

7.° quia igitur (9)

$$f(z)\frac{d\left[\frac{\chi(z)}{z-x}\right]}{dz}=\frac{d\left[\frac{f(z)\chi(z)}{z-x}\right]}{dz}-f'(z)\frac{\chi(z)}{z-x},$$

$$f(z)\frac{d\left[\frac{\chi(z)}{(z-x)^n}\right]}{dz}=\frac{d\left[\frac{f(z)\chi(z)}{(z-x)^n}\right]}{dz}-f'(z)\frac{\chi(z)}{(z-x)^n},$$

iccirco residua functionum

$$f(z)\frac{d\left[\frac{\chi(z)}{z-x}\right]}{dz},\quad f(z)\frac{d\left[\frac{\chi(z)}{(z-x)^n}\right]}{dz}$$

quoad $z=x$ erunt (6.°)

$$\mathcal{E}\frac{f(z)}{(x-x_1)^n((\,[z-x_1]^n\,))}[(z-x_1)^{n-1}+$$

$$(x-x_1)(z-x_1)^{n-2}+(x-x_1)^2(z-x_1)^{n-3}+\ldots+$$

$$(x-x_1)^{n-1}]=$$

$$\mathcal{E}\frac{f(z)}{(x-x_1)^n((\,[z-x_1]^n\,))}\cdot\frac{(x-x_1)^n-(z-x_1)^n}{x-z}=$$

$$\mathcal{E}\frac{f(z)}{(x-z)((\,[z-x_1]^n\,))}-\frac{1}{(x-x_1)^n}\mathcal{E}\frac{(z-x_1)^n f(z)}{(x-z)((\,[z-x_1]^n\,))},$$

ubi

$$\mathcal{E}\frac{(z-x_1)^n f(z)}{(x-z)((\,[z-x_1]^n\,))},\ \text{seu}\ \mathcal{E}\frac{(z-x_1)f(z)}{(x-z)((\,z-x_1\,))}$$

exhibet residuum functionis

$$\frac{f(z)}{x-z}$$

relate ad $z=x_1$, quod residuum (67. 2.°) $=0$: igitur

$$\frac{f(x_1)}{(x-x_1)^n}+\frac{1}{1}\frac{f'(x_1)}{(x-x_1)^{n-1}}+$$

$$\frac{1}{1.2}\frac{f''(x_1)}{(x-x_1)^{n-2}}+\ldots+\frac{1}{1.2.3\ldots(n-1)}\frac{f^{(n-1)}(x_1)}{x-x_1}=$$

$$\mathcal{E}\frac{f(z)}{(x-z)((\,[z-x_1]^n\,))}.$$

Fiat

$$\frac{f^{(n)}(x_1+\varepsilon(x-x_1))}{1.2.3\ldots n}=\psi(x)\ldots(g^{VII});$$

proveniet

$$f(x) = \mathcal{E}\frac{f(z)}{(x-z)\,(([z-x_1]^n))} + \psi(x)\,;$$

ideoque (65. g''')

$$f(x) - \mathcal{E}\frac{(z-x_1)f(z)}{(x-z)\,((z-x_1))} = \psi(x)\,\ldots\,(g^{\mathrm{VIII}}).$$

69. Haec nunc facile stabiliuntur. 1.° sumpta $x=x_1$, erit (68. g^{VII})

$$\psi(x_1) = \frac{f^{(n)}(x_1)}{1.2.3\ldots n}\,;$$

functio videlicet $\psi(x_1)$ valorem obtinet finitum: proinde (68. g^{VIII}) etsi $f(x)$ evadit infinita quando $x=x_1$, tamen differentia inter $f(x_1)$ et residuum functionis

$$\frac{f(z)}{x-z}$$

relate ad $z=x_1$ manet quantitas finita sub eadem positione $x=x_1$. 2.° denotante igitur

$$\mathcal{E}\frac{((f(z)))}{x-z}$$

summam residuorum functionis

$$\frac{f(z)}{x-z}$$

quoad $z=x_1$, $z=x_2$, $z=x_3$, ... quantitas

$$f(x) - \mathcal{E}\frac{((f(z)))}{x-z}$$

nunquam evadet infinita, quaecumque caeteroquin e

radicibus x_1, x_2, x_3, ... aequationis (g) adhibeatur pro x. Quare si fiat

$$f(x) - \mathcal{E}\frac{((f(z)))}{x-z} = F(x) \ldots (g^{IX}),$$

functio $F(x)$ permanebit finita quoad $x = x_1$, $x = x_2$, $x = x_3$, ..., et consequenter (67. 2°.) quoad omnes quantitatis variabilis x valores finitos. 3.° si $f(x)$ est fractio rationalis, erit $F(x)$ ejusmodi fractio similiter rationalis in qua denominator nunquam fiet $= 0$; habebit videlicet functio $F(x)$ denominatorem constantem, eritque proinde $F(x)$ functio integra quantitatis variabilis x. 4°. quibus positis, assumatur

$$f(x) = \frac{\chi(x)}{\varphi(x)},$$

ubi $\chi(x)$, $\varphi(x)$ designant functiones integras variabilis x, ponaturque gradus denominatoris $\varphi(x)$ major quam gradus numeratoris $\chi(x)$: existet

$$f(x) = 0; \text{ si } x = \infty,$$

totumque (67. 2.°) primum membrum aequationis (g^{IX}) evanescet, ideoque et secundum: quod cum sit functio integra quantitatis x, oportet in ea qua sumus hypothesi existat constanter $F(x) = 0$, et consequenter

$$f(x) = \mathcal{E}\frac{((f(z)))}{x-z} \ldots (g^{X}).$$

5.° multiplicetur (g^{X}) per x, ut prodeat

$$x f(x) = \mathcal{E}\frac{((f(z)))}{1 - \frac{z}{x}}:$$

facta $x = \infty$, exhibeatur valor $x f(x)$ per Θ; erit

$$\left.\begin{array}{r}\Theta = \mathcal{E}\,((f(z)));\\ \text{et evanescente } \Theta\,, \qquad\qquad\\ \mathcal{E}\,((f(z))) = 0\,;\end{array}\right\}(g^{xi})$$

quarum secunda exposcit ut in fractione rationali

$$f(x) = \frac{\chi(x)}{\varphi(x)}$$

non modo gradus denominatoris superet numeratoris gradum, sed etiam differentia inter utrumque gradum sit > 1.

70. Data functione $f(x, y)$ binarum varibilium independentium x et y, ponatur aequatio

$$\frac{1}{f(x, y)} = 0$$

resoluta quoad x suppeditare radices $x_1, x_2, \ldots$; ac 1.° sit (65. g'')

$$f(x_1 + \theta, y + \theta') = \frac{\mathrm{f}(x_1 + \theta, y + \theta')}{\theta}\;;$$

erit (63. o'': 66. g^{v})

$$\mathcal{E}\,\frac{(x - x_1)\,f(x, y)}{((x - x_1))} = \mathrm{f}(x_1, y),$$

unde

$$\frac{d_y\,\mathcal{E}\,\dfrac{(x - x_1)\,f(x, y)}{((x - x_1))}}{dy} = \frac{d_y\,\mathrm{f}(x_1, y)}{dy}\;;$$

item (65. g')

$$\frac{d_y f(x, y)}{dy} = \frac{\dfrac{d_y \mathrm{f}(x, y)}{dy}}{x - x_1},$$

et consequenter (67. 4.°)

$$\mathcal{E}\frac{(x-x_1)\dfrac{d_y f(x\,,\,y)}{dy}}{((x-x_1))}=\frac{d_y \mathrm{f}(x\,,\,y)}{dy}\,,$$

2.° sit (65. g^{IV})

$$f(x_1+\theta\,,\,y+\theta')=\frac{\mathrm{f}(x_1+\theta\,,\,y+\theta')}{\theta^n}\,:$$

erit (63. o'' : 66. g^{VI})

$$\mathcal{E}\frac{(x-x_1)f(x\,,\,y)}{((x-x_1))}=\frac{1}{1.2\ldots(n-1)}\cdot\frac{d^{n-1}\mathrm{f}(x_1\,,\,y)}{dx_1^{\,n-1}}\,,$$

ac proinde

$$\frac{d_y\,\mathcal{E}\dfrac{(x-x_1)f(x\,,\,y)}{((x-x_1))}}{dy}=\frac{d^n\mathrm{f}(x_1\,,\,y)}{dx_1^{\,n-1}dy}\,;$$

item (65. g''')

$$\frac{d_y f(x\,,\,y)}{dy}=\frac{\dfrac{d_y \mathrm{f}(x\,,\,y)}{dy}}{(x-x_1)^n}\,,$$

ideoque (67. 5.°)

$$\mathcal{E}\frac{(x-x_1)\dfrac{d_y f(x\,,\,y)}{dy}}{((x-x_1))}=\frac{d^n\mathrm{f}(x_1\,,\,y)}{dx_1^{\,n-1}dy}\,.$$

Non pluribus opus est ut intelligamus fore

$$\frac{d_y\,\mathcal{E}\dfrac{(x-x_1)f(x\,,\,y)}{((x-x_1))}}{dy}=\mathcal{E}\frac{(x-x_1)\dfrac{d_y f(x\,,\,y)}{dy}}{((x-x_1))}\,:$$

simili modo

$$\frac{d_y \mathcal{E} \frac{(x-x_2)f(x,y)}{((x-x_2))}}{dy} = \mathcal{E} \frac{(x-x_2)\frac{d_y f(x,y)}{dy}}{((x-x_2))},$$

atque ita porro. Hinc

$$\frac{d_y \mathcal{E}((f(x,y)))}{dy} = \mathcal{E}\left(\left(\frac{d_y f(x,y)}{dy}\right)\right) \ldots (g^{\mathrm{XIII}}).$$

71. Ope formulae (g^{X}) resolvuntur fractiones rationales in alias simpliciores: rem bina declarabunt exempla.

1.° Detur fractio

$$\frac{x^3+x^2+2}{x(x-1)^2(x+1)^2}:$$

erit

$$\frac{x^3+x^2+2}{x(x-1)^2(x+1)^2} = \mathcal{E}\left(\left(\frac{z^3+z^2+2}{z(z-1)^2(z+1)^2}\right)\right)\frac{1}{x-z} =$$

$$\mathcal{E}\frac{z^3+z^2+2}{((z))(z-1)^2(z+1)^2(x-z)} + \mathcal{E}\frac{z^3+z^2+2}{z(([z-1]^2))(z+1)^2(x-z)} +$$

$$\mathcal{E}\frac{z^3+z^2+2}{z(z-1)^2(([z+1]^2))(x-z)}.$$

Jam vero relate ad $z=x_1=0$ est (66. g^{V})

$$\mathcal{E}\frac{z^3+z^2+2}{((z))(z-1)^2(z+1)^2(x-z)} = f(x_1) = \frac{2}{x};$$

insuper quoad $z=x_2=1$ habemus (66. g^{VI}: 67. 4.°)

$$\mathcal{E}\frac{z^3+z^2+2}{((z))(z-1)^2\ (z+1)^2(x-z)} = f(x_1) = \frac{2}{x};$$

insuper quoad $z = x_2 = 1$ habemus (66. g^{vi}: 67. 4.°)

$$\mathcal{E}\,\frac{z^3+z^2+2}{z((\,[z-1]^2)\,)\,(z+1)^2(x-z)} = f'(x_2) =$$

$$\frac{d[\theta^2 f(x_2+\theta)]}{d\theta} = \frac{d\,\dfrac{(1+\theta)^3+(1+\theta)^2+2}{(1+\theta)\,(2+\theta)^2(x-1-\theta)}}{d\theta} =$$

$$\frac{1}{(x-1)^2} - \frac{3}{4(x-1)}\,;$$

denique quoad $z = x_3 = -1$,

$$\mathcal{E}\,\frac{z^3+z^2+2}{z(z-1)^2((\,[z+1]^2)\,)\,(x-z)} =$$

$$\frac{d\,\dfrac{(\theta-1)^3+(\theta-1)^2+2}{(\theta-1)\,(\theta-2)^2(x+1-\theta)}}{d\theta} = -\frac{1}{2(x+1)^2} - \frac{5}{4(x+1)}\,:$$

igitur

$$\frac{x^3+x^2+2}{x(x-1)^2(x+1)^2} = \frac{2}{x} + \frac{1}{(x-1)^2} - \frac{3}{4(x-1)} -$$

$$\frac{1}{2(x+1)^2} - \frac{5}{4(x+1)}\,.$$

II.° Detur fractio

$$\frac{1}{x^3(x^2+1)}\,:$$

erit

$$\frac{1}{x^3(x^2+1)} = \mathcal{E}\,\frac{1}{((z^3))\,(z^2+1)(x-z)} +$$

$$\mathcal{E}\frac{1}{z^3((z^2+1)).(x-z)}=\mathcal{E}\frac{1}{((z^3))(z^2+1)(x-z)}+$$

$$\mathcal{E}\frac{1}{z^3((z-\sqrt{-1}))(z+\sqrt{-1})(x-z)}+$$

$$\mathcal{E}\frac{1}{z^3(z-\sqrt{-1})((z+\sqrt{-1}))(x-z)}.$$

Habemus autem quoad $z=x_1=0$

$$\mathcal{E}\frac{1}{((z^3))(z^2+1)(x-z)}=\frac{1}{2}\frac{d^2\frac{1}{(\theta^2+1)(x-\theta)}}{d\theta^2}=$$

$$\frac{1}{x^3}-\frac{1}{x};$$

praeterea quoad $z=x_2=\sqrt{-1}$,

$$\mathcal{E}\frac{1}{z^3((z-\sqrt{-1}))(z+\sqrt{-1})(x-z)}=$$

$$\frac{1}{(\sqrt{-1})^3 \, . \, 2\sqrt{-1}(x-\sqrt{-1})}=\frac{1}{2(x-\sqrt{-1})};$$

tandem quoad $z=x_3=-\sqrt{-1}$,

$$\mathcal{E}\frac{1}{z^3(z-\sqrt{-1})((z+\sqrt{-1}))(x-z)}=$$

$$\frac{1}{-(-\sqrt{-1})^3 \, . \, 2\sqrt{-1}(x+\sqrt{-1})}=\frac{1}{2(x+\sqrt{-1})};$$

ergo $\dfrac{1}{x^2(x^2+1)}=\dfrac{1}{x^2}-\dfrac{1}{x}+\dfrac{1}{2(x-\sqrt{-1})}+$

$$\frac{1}{2(x+\sqrt{-1})}.$$

72. Denotante z_1 unam e radicibus aequationis

$$z-x-k\varphi(z)=0\ldots(g^{xlii}),$$

proponitur evolvenda $f(z_1)$ in seriem ordinatam per potentias ascendentes quantitatis k.

Habemus (67. 3.°)

$$\mathcal{E}\frac{(z-z_1)f(z)}{((z-z_1))(z-x-k\varphi(z))}=\frac{f(z_1)}{1-k\varphi'(z_1)};$$

et consequenter

$$f(z_1)=\mathcal{E}\frac{(z-z_1)[1-k\varphi'(z)]f(z)}{((z-z_1))(z-x-k\varphi(z))}.$$

Est autem (23. III.°)

$$L(z-x-k\varphi(z))=L(z-x)-\frac{k\varphi(z)}{z-x}-\frac{k^2}{2}[\frac{\varphi(z)}{z-x}]^2-\ldots$$

$$-\frac{k^n}{n}[\frac{\varphi(z)}{z-x}]^n-\frac{k^{n+1}}{n+1}[\frac{\varphi(z)}{z-x-\varepsilon k\varphi(z)}]^{n+1};$$

ac proinde (6. 6° : 7 : 8), facto compendii causa

$$-\frac{k^{n+1}}{n+1}[\frac{\varphi(z)}{z-x-\varepsilon k\varphi(z)}]^{n+1}=F,$$

$$1.° \frac{1-k\varphi'(z)}{z-x-k\varphi(z)}=\frac{1}{z-x}-k\frac{d[\frac{\varphi(z)}{z-x}]}{dz}-$$

$$\frac{k^2}{2}\,\frac{d\left[\frac{\varphi(z)}{z-x}\right]^2}{dz}-\dots-\frac{k^n}{n}\,\frac{d\left[\frac{\varphi(z)}{z-x}\right]^n}{dz}+F'_z=$$

$$\frac{1}{z-x}\left[1+\frac{k\varphi(z)}{z-x}+\left(\frac{k\varphi(z)}{z-x}\right)^2+\dots\right.$$

$$\left.+\left(\frac{k\varphi(z)}{z-x}\right)^n\right]-\frac{k\varphi'(z)}{z-x}\left[1+\frac{k\varphi(z)}{z-x}+\dots\right.$$

$$\left.+\left(\frac{k\varphi(z)}{z-x}\right)^{n-1}\right]+F'_z=\frac{1}{z-x}\left[\frac{1-\left(\frac{k\varphi(z)}{z-x}\right)^{n+1}}{1-\frac{k\varphi(z)}{z-x}}\right]-$$

$$\frac{k\varphi'(z)}{z-x}\left[\frac{1-\left(\frac{k\varphi(z)}{z-x}\right)^{n}}{1-\frac{k\varphi(z)}{z-x}}\right]+F'_z;$$

2.ª $1-k\varphi'(z)=1-\left(\frac{k\varphi(z)}{z-x}\right)^{n+1}-k\varphi'(z)+$

$$k\varphi'(z)\left(\frac{k\varphi(z)}{z-x}\right)^n+\left(z-x-k\varphi(z)\right)F'_z;$$

3.ª $F'_z=\dfrac{k^{n+1}[\varphi(z)]^n\,[\varphi(z)-(z-x)\varphi'(z)]}{(z-x)^{n+1}\,[z-x-k\varphi(z)]};$

4.ª $\dfrac{1-k\varphi'(z)}{z-x-k\varphi(z)}f(z)=\dfrac{f(z)}{z-x}-\dfrac{k}{1}f(z)\dfrac{d\left[\frac{\varphi(z)}{z-x}\right]}{dz}-$

$$\frac{k^2}{2}f(z)\frac{d\left[\frac{\varphi(z)}{z-x}\right]^2}{dz}-\dots-\frac{k^n}{n}f(z)\,\frac{d\left[\frac{\varphi(z)}{z-x}\right]^n}{dz}+$$

$$f(z)\frac{k^{n+1}[\varphi(z)]^n[\varphi(z)-(z-x)\varphi'(z)]}{(z-x)^{n+1}[z-x-k\varphi(z)]}.$$

Extractis itaque residuis ex 4.ª quoad solas $z=z_1$, $z=x$, et posito compendii causa

$$f(z)\frac{k^{n+1}[\varphi(z)]^n[\varphi(z)-(z-x)\varphi'(z)]}{(z-x)^{n+1}[z-x-k\varphi(z)]}=\psi \ldots (g^{XIV}),$$

proveniet (67. 5°. 7°. 8°.)

$$\left.\begin{aligned} f(z_1)=f(x)+\frac{k}{1}f'(x)\varphi(x)+\frac{k^2}{1.2}\frac{d[f'(x)(\varphi(x))^2]}{dx}+\ldots \\ +\frac{k^n}{1.2\ldots n}\frac{d^{n-1}[f'(x)(\varphi(x))^2]}{dx^{n-1}}+\mathcal{E}\frac{(z-x)(z-z_1)\psi}{(([z-x][z-z_1]))}. \end{aligned}\right\}(g^{XV})$$

Si, aucto n indefinite, ponitur

$$lim. \mathcal{E}\frac{(z-x)(z-z_1)\psi}{(([z-x][z-z_1]))}=0,$$

valebit aequatio

$$\left.\begin{aligned} f(z_1)=f(x)+\frac{k}{1}f'(x)\varphi(x)+\frac{k^2}{1.2}\frac{d[f'(x)(\varphi(x))^2]}{dx}+ \\ \frac{k^3}{1.2.3}\frac{d^2[f'(x)(\varphi(x))^3]}{dx^2}+\ldots: \end{aligned}\right\}(g^{XV})$$

huc spectat series Lagrangiana.

CALCULI DIFFERENTIALIS AD GEOMETRIAM APPLICATIO.

LINEAE IN SUPERFICIE PLANA CONSTITUTAE.

De tangentibus, subtangentibus, normalibus, subnormalibus, et asymptotis.

73. Sit curva (5)

$$y = f(x) \dots (\mathrm{o})$$

ad coordinatas orthogonales relata, et per (x, y) exhibeatur quodvis curvae punctum coordinatis x, y determinatum; sit (174 ex p. 2.^{a}) τ tangens, p subtangens, ν normalis, q subnormalis, c chorda respondens infinitesimo arcus s incremento Δs computato a contactus puncto (x, y): designatis angulis ut in p. 2.^{a} n. 171, erit (125 ex p. 2.^{a})

$$tang.\,(cx) = \frac{\Delta y}{\Delta x}.$$

Sed vergente Δs ad $lim. = 0$, vergit angulus (cx) ad angulum (τx); igitur (6)

$$\left.\begin{array}{l} tang.\,(\tau x) = lim.\,\dfrac{\Delta y}{\Delta x} = \dfrac{dy}{dx}. \\ \text{Exinde (125 : 45. 1° : 40. 1.° ex p. 2.}^a\text{)} \\ p = \dfrac{y}{tang(\tau x)} = \dfrac{y\,dx}{dy},\ \tau = \sqrt{y^2 + p^2} = y\sqrt{\left[1 + \dfrac{dx^2}{dy^2}\right]}, \\ \nu = \tau\,tang(\tau x) = y\sqrt{\left[1 + \dfrac{dy^2}{dx^2}\right]},\ q = \dfrac{y^2}{p} = \dfrac{y\,dy}{dx}. \end{array}\right\} (\mathrm{o}')$$

Ad haec: denotantibus v et u coordinatas tangentis vel normalis, erit (172. I.° ex p. 2.ª)

$$\left.\begin{array}{l} u-y=\frac{dy}{dx}(v-x) \text{ aequatio ad tangentem}, \\ \text{et} \\ u-y=-\frac{dx}{dy}(v-x) \text{ aequatio ad normalem}. \end{array}\right\} (o'')$$

Exempla.

I.° Aequatio ad cycloidem jam inventa in p. 2.ª n. 172. III.° suppeditat (11)

$$dy=\frac{a.d\frac{\sqrt{2ax-x^2}}{a}}{\sqrt{1-\frac{2ax-x^2}{a^2}}}+d\sqrt{2ax-x^2}=$$

$$\frac{2a-x}{a-x}d\sqrt{2ax-x^2};$$

ideoque (6. 4° : 7)

$$dy=dx\sqrt{[\frac{2a-x}{x}]}\ldots(o''');$$

coordinatae computantur a supremo axeos puncto seu a cycloidis *vertice*: quod si computandae sint a baseos initio ita ut abscissae sumantur in ipsa basi, satis erit in (o''') substituere (171 ex p. 2.ª) $2a-y$ loco x et $a\pi-x$ loco y; quae substitutiones praebent

$$dy=dx\sqrt{[\frac{2a-y}{y}]}\ldots(o^{IV}).$$

Inhaerentes aequationi (o^{IV}) assequemur

$$tang(\tau x) = \sqrt{[\frac{2a-y}{y}]},\ p = y\sqrt{[\frac{y}{2a-y}]},$$

$$\tau = y\sqrt{[\frac{2a}{2a-y}]},\ \nu = \sqrt{2ay},\ q = \sqrt{2ay-y^2}.$$

Facile intelligitur illud: normalis in cycloide nihil est aliud (52. 3.° ex p. 2.ª) nisi chorda subtendens arcum circuli genitoris interceptum cycloidis basi et ipsius cycloidis puncto (x, y); hinc sequitur (52. 1.° ex p. 2.ª) supplementum illius arcus computatum a puncto (x, y) extra cycloidem subtendi chorda, quae in puncto (x, y) cycloidem tangit.

II.° Invenire subtangentem in Logarithmica (173. IV.° ex p. 2.ª).

Erit (6. 5.°) $dy = \frac{L(a)}{c} ba^{\frac{x}{c}} dx$, ideoque subtangens quaesita

$$p = \frac{ydx}{dy} = \frac{c}{L(a)};$$

ubique nimirum eadem.

Notetur illud: duae curvae (o) et $y = f(x)$ ibi dicuntur se mutuo tangere ubi communem habent tangentem; hinc quoad ejusmodi contactus punctum ob primam (o') exsistent simul

$$f(x) = \mathrm{f}(x),\ f'(x) = \mathrm{f}'(x).$$

74. Si in prima (o'') ponitur x indefinite crescere, facto ad limites gradu poterit sciri utrum curva (o) gaudeat asymptotis.

Exempla.

I.° Quoad logarithmicam

$$y = e^x$$

habemus $u - e^x = e^x(v - x)$; et facta $x = -\infty$, emerget (241. 1.° ex p. 1.^{a})

$$u = 0:$$

asymptotus nempe recidit in abscissarum axem.

II.° Quoad parabolam

$$y^2 = px$$

assequimur $u \mp \sqrt{px} = \pm \frac{1}{2}(v - x)\sqrt{\frac{p}{x}}$, $u =$

$$\pm \tfrac{1}{2}\left(\sqrt{px} + v\sqrt{\frac{p}{x}}\right);$$ et facta $x = \infty$,

$$u = \pm \infty:$$

parabola videlicet caret asymptotis.

III.° Quoad hyperbolam

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

habemus $u \mp b\left(\frac{x^2}{a^2} - 1\right)^{\frac{1}{2}} = \pm \frac{bx(v-x)}{a^2\left(\frac{x^2}{a^2} - 1\right)^{\frac{1}{2}}}$,

$$\text{seu } u\left(\frac{x^2}{a^2} - 1\right)^{\frac{1}{2}} \pm b = \pm \frac{bxv}{a^2};$$

quae multiplicata per $\frac{a}{x}$ vertitur in

$$u\left(1 - \frac{a^2}{x^2}\right)^{\frac{1}{2}} \pm b\,\frac{a}{x} = \pm \frac{b}{a}\,v:$$

et facta $x = \pm \infty$, prodibit aequatio

$$u = \pm \frac{b}{a} v$$

ad hyperbolae asymptotos. Recole p. 2^{ae}, n. 173. IV.° : 201. 1.°

75. Si $tang(\tau x)$, p, τ, ν, q debeant exprimi per coordinatas polares z et ω (174 ex p. 2.^{a}), praeter valorem y praesto quoque erunt valores

$$dx = \cos\omega dz - z\sin\omega d\omega, \quad dy = \sin\omega dz + z\cos\omega d\omega$$

substituendi in (o) : recta z dicitur *radius vector*; punctum vero, circa quod revolvitur z, *polus*.

Exemplum.

Proponatur aequatio

$$z = \frac{\omega}{2\pi} \ldots (o^{v}),$$

in qua z crescit vel decrescit in eadem ratione ac arcus ω; fiet $z = 0$ si $\omega = 0$, et exinde ambo poterunt simul in infinitum augeri : curva itaque (o^{v}) incipiendo a polo, atque inde magis semper recedendo sese revolvet sine fine circa ipsum polum. Jam si quaeratur subtangens p, erit

$$p = \frac{ydx}{dy} = z\sin\omega\frac{\cos\omega dz - z\sin\omega d\omega}{\sin\omega dz + z\cos\omega d\omega} :$$

pone $\omega = \frac{\pi}{2}$, ut a praefato puncto computetur subtangens in recta ad perpendiculum insistente radio vectori z; habebis $p = -\frac{z^2 d\omega}{dz}$, seu ob ($o^{v}$)

$$p = -2\pi z^2 = -\frac{\omega^2}{2\pi}.$$

Facto $\omega = 2n\pi$ (denotat n numerum revolutionum rectae z), proveniet subtangens $p = -m.2n\pi$; aequalis nimirum peripheriae circuli habentis radium $= n.1$ ductae in revolutionum numerum. Curva (o^v), quam Conon Syracusanus excogitavit, est una ex illis quae dicuntur *spirales*: ejus praecipuas proprietates investigavit Archimedes.

De differentialibus arcus et areae curvilineae.

76. Infinitesimo arcus incremento Δs vergente ad $lim. = 0$, ratio inter Δs et respondentem chordam c vergit ad $lim. = 1$. Per punctum namque (x, y) intelligatur duci tangens, cui occurret ordinata $y + \Delta y$ producta si opus est: designante k eam tangentis portionem quae intercipitur puncto contactus et ordinata $y + \Delta y$ ...., h eam ordinatae $y + \Delta y$ .... portionem quae intercipitur tangente et extremitate arcus $s + \Delta s$, erit (73)

$$\Delta y + h = \Delta x . tang(\tau x) = \Delta x . \frac{dy}{dx}$$

si curva obvertit cavitatem axi abscissarum, et

$$\Delta y - h = \Delta x . tang(\tau x) = \Delta x . \frac{dy}{dx}$$

si convexitatem; ideoque

$$h = \pm (\Delta x . \frac{dy}{dx} - \Delta y),$$

$$k = \sqrt{(\Delta y \pm h)^2 + \Delta x^2} = \Delta x \sqrt{[1 + \frac{dy^2}{dx^2}]};$$

item

$$c = \sqrt{\Delta x^2 + \Delta y^2} = \Delta x \sqrt{[1 + \frac{\Delta y^2}{\Delta x^2}]}:$$

hinc

$$\frac{h+k}{c} = \frac{\pm(\frac{dy}{dx} - \frac{\Delta y}{\Delta x}) + \sqrt{[1 + \frac{dy^2}{dx^2}]}}{\sqrt{[1 + \frac{\Delta y^2}{\Delta x^2}]}};$$

et consequenter

$$lim.\frac{h+kc}{c} = \frac{\sqrt{[1+\frac{dy^2}{dx^2}]}}{\sqrt{[1+\frac{dy^2}{dx^2}]}} = 1.$$

Atqui

$$\frac{\Delta s}{c} > \frac{c}{c} = 1, \text{ simulque } \frac{\Delta s}{c} < \frac{h+k}{c}:$$

ergo

$$lim.\frac{\Delta s}{c} = 1.$$

77. Cum igitur sit

$$lim.\frac{\Delta s}{c} = lim.\frac{\Delta s}{\Delta x\sqrt{[1+\frac{\Delta y^2}{\Delta x^2}]}} = 1,$$

cumque

$$lim.\frac{\Delta s}{\Delta x} = \frac{ds}{dx}, \; lim.\frac{\Delta y^2}{\Delta x^2} = \frac{dy^2}{dx^2},$$

exsistet igitur

$$\frac{ds}{dx\sqrt{\left[1+\frac{dy^2}{dx^2}\right]}}=1;$$

ac proinde differentiale arcus s sic exprimetur

$$ds=dx\sqrt{\left[1+\frac{dy^2}{dx^2}\right]}=\sqrt{dx^2+dy^2}.$$

Substitutis valoribus dx, dy ex (75), prodibit

$$ds=\sqrt{dz^2+z^2d\omega^2}.$$

78. Inquiramus nunc in differentiale areae α interceptae arcu s et rectis lineis y_0, y, $x-x_0$.

Denotantibus t, t' trapezia, alterum interceptum lateribus (76)

$$k,\ y,\ y+\Delta y\pm h,\ \Delta x,$$

alterum lateribus

$$c,\ y,\ y+\Delta y,\ \Delta x,$$

erunt (44. 5.° ex p. 2.[a])

$$t=\frac{\Delta x}{2}(y+y+\Delta y\pm h)=\frac{\Delta x}{2}\left(2y\pm\Delta x\frac{dy}{dx}\right),$$

$$t'=\frac{\Delta x}{2}(y+y+\Delta y)=\frac{\Delta x}{2}(2y+\Delta y);$$

unde

$$lim.\frac{t}{\Delta x}=y,\quad lim.\frac{t'}{\Delta x}=y.$$

Sed ratio $\frac{\Delta\alpha}{\Delta x}$ interjacet rationes $\frac{\Delta t}{\Delta x}$, $\frac{\Delta t'}{\Delta x}$; igitur

$$lim.\frac{\Delta\alpha}{\Delta x}=y,\quad \frac{d\alpha}{dx}=y,\quad d\alpha=ydx.$$

79. Determinandum quoque sit differentiale sectoris α' intercepti arcu s et radiis vectoribus ab origine coordinatarum ductis ad puncta (x_0, y_0), (x, y). Quoniam (44. 4.° ex p. 2.ª)

$$\alpha' = \alpha + \frac{x_0 y_0}{2} - \frac{xy}{2},$$

ideo

$$d\alpha' = d\alpha - d\frac{xy}{2}, \text{ seu (78)}$$

$$d\alpha' = y dx - d\frac{xy}{2} = \frac{y dx - x dy}{2}.$$

De punctis inflexionis.

80. Sic appellantur puncta illa in quibus desinit curva obvertere cavitatem suam in partes v. gr. abscissarum ut ultra progrediens in easdem obvertat convexitatem. Antequam determinemus ejusmodi puncta, praemittimus illud: quaeritur utrum apud punctum (x, y) curva (o) obvertat cavitatem vel convexitatem axi abscissarum.

In casu cavitatis obversae (76)

$$\Delta y + h = \Delta x \frac{dy}{dx}; \text{ ideoque } \Delta x \frac{dy}{dx} > \Delta y, \; \frac{dy}{dx} > \frac{\Delta y}{\Delta x}:$$

in casu convexitatis obversae

$$\Delta y - h = \Delta x \frac{dy}{dx}; \text{ ac proinde } \Delta x \frac{dy}{dx} < \Delta y, \; \frac{dy}{dx} < \frac{\Delta y}{\Delta x}.$$

Sed in utroque casu

$$\frac{dy}{dx} - lim. \frac{\Delta y}{\Delta x} = \frac{dy}{dx} - \frac{dy}{dx} = 0;$$

ergo, accedente puncto $(x + \Delta x, y + \Delta y)$ ad punctum

(x, y), crescet $\frac{\Delta y}{\Delta x}$ in 1.° casu, decrescet in 2.°; ideoque (17) in puncto (x, y)

$$\frac{d\frac{\Delta y}{\Delta x}}{dx} < 0 \text{ in 1.° casu}, > 0 \text{ in 2.°}$$

Atqui in puncto (x, y) vertitur $\frac{\Delta y}{\Delta x}$ in $lim.\frac{\Delta y}{\Delta x} = \frac{dy}{dx}$; igitur curva apud punctum (x, y) obvertet cavitatem, vel convexitatem abscissarum axi, prout

$$\frac{d^2y}{dx^2} <, \text{ vel } > 0.$$

81. Quibus positis, cum, praetergressis inflexionis punctis, debeat $f''(x)$ fieri $>$ vel < 0, quae, priusquam ad ipsa perveniretur, erat $<$ vel > 0, propterea in illis punctis derivata $f''(x)$ vel evanescet, vel evadet infinita (131. 1.° ex p. 1.[a]); ideoque abscissae x_n, quibus respondere potest inflexio, erunt quaerendae inter radices aequationum

$$f''(x) = 0, \quad \frac{1}{f''(x)} = 0.$$

Ut autem abscissis hoc pacto definitis revera inflexio respondeat, debet insuper signum derivatae $f''(x)$ mutari si, loco x_n subrogato $x_n + \sigma$, infinitesima σ ex $>$ vel < 0 sumatur $<$ vel > 0.

Exemplum.

Sit curva

$$y = a + \frac{x(a^2 - x^2)\sqrt{a^2 - x^2}}{a^3}:$$

erunt

$$f'(x)=\frac{(a^2-4x^2)\sqrt{a^2-x^2}}{a^3},\ f''(x)=\frac{12x^3-9a^2x}{a^3\sqrt{a^2-x^2}}.$$

Aequatio $f''(x)=0$ praebet

$$x_n=0,\ x_n=\frac{a\sqrt{3}}{2},\ x_n=-\frac{a\sqrt{3}}{2}:$$

spectato primo valore x_n, habemus

$$f''(x_n+\sigma)=\frac{12\sigma^3-9a^2\sigma}{a^3\sqrt{a^2-\sigma^2}};$$

et spectatis duobus aliis,

$$f''(x_n+\sigma)=\frac{12\sigma^3\pm18a\sigma^2\sqrt{3}+18a^2\sigma}{a^3\sqrt{[a^2-(\sigma\pm\frac{a\sqrt{3}}{2})^2]}};$$

quoniam igitur primus valor $f''(x_n+\sigma)$ ex $<$ fit >0 (251 ex p. 1.[a]), et alii duo ex $>$ fiunt <0 quando σ ex $>$ accipitur <0, iccirco ternis abscissis x_n totidem respondebunt inflexionis puncta. Quia vero quoad primum punctum est $f'(x_n)=1$, et quoad alia duo $f'(x_n)=-1$, ideo (73) tangens ducta per haec tria puncta occurret abscissarum axi sub angulo semirecto, etsi non ad eamdem partem semper obverso.

82. Quoad inflexionem respondentem radici x_n aequationis $f''(x)=0$ haec notentur 1.° facile admodum stabilitur (21) formula

$$f''(x_n+\sigma)=\frac{\sigma^m}{1.2...m}f^{(m)}(x_n+\varepsilon\sigma);$$

et quia $f^{(m)}(x_n+\varepsilon\sigma)$ in viciniis x_n eodem afficitur signo ac $f^{(m)}(x_n)$, propterea, mutato signo infinitesi-

mae σ, haud immutabitur signum derivatae $f''(x_n+\sigma)$ nisi m fuerit impar: radici (81) nimirum x_n respondebit inflexio quotiescumque ex derivatis

$$f'''(x_n),\ f^{\mathrm{IV}}(x_n),\ f^{\mathrm{V}}(x_n),\ \ldots$$

quae non evanescit prima, erit ordinis imparis. 2.° Exinde infertur (30 : 31) $f'(x)$ seu (73) $tang(\tau x)$ fore maximam minimamve apud inflexionis punctum (x_n, y_n).

De circulo osculatore, deque evolutis; ubi et aliquid generatim annotatur circa curvas osculatrices.

83. Haec praemittimus: 1.° circuli peripheria eo magis vel minus apud contactus punctum accedit ad tangentem, quo major vel minor (52. 13.° ex p. 2.ª) fit ejus radius; hinc circulorum curvaturae censentur esse reciproce ut respondentes radii. 2.° duobus punctis (x, y), $(x+\Delta x, y+\Delta y)$ intelligatur intercipi circularis arcus $\Delta\chi$, cujus radius $=\rho$, tangens geometrica $=\theta'$ in puncto (x, y), $=\theta''$ in puncto $(x+\Delta x, y+\Delta y)$: angulus quem faciunt θ' et θ'' aequatur (49 : 35. 7° : 7 ex p. 2.ª) angulo, quem continent bini radii pertingentes ad puncta illa, et consequenter (61 : 124 ex p. 2.ª)

$$\Delta\chi=\rho(\theta'\theta''),\quad \frac{1}{\rho}=\frac{(\theta'\theta'')}{\Delta\chi}.$$

Est autem (35. 8.° ex p. 2.ª) $(\theta'\theta'')=(\theta' x)-(\theta'' x)=\Delta(\theta'' x)$; igitur

$$\frac{1}{\rho}=\frac{\Delta(\theta'' x)}{\Delta\chi}.$$

84. Quibus positis, fac ut puncta (x, y), $(x+\Delta x, y+\Delta y)$ praeter circularem arcum $\Delta\chi$ intercipiant quoque arcum Δs curvae (o): accedente $(x+\Delta x, y+\Delta y)$

ad (x, y), vergunt (76) $\Delta\chi$, Δs ad communem chordam $\sqrt{\Delta x^2 + \Delta y^2}$; ac proinde ad mutuam aequalitatem, ut sit

$$lim. \frac{\Delta\chi}{\Delta s} = \frac{d\chi}{ds} = 1:$$

inferimus radium ρ ita immutari, ut curvatura circuli vergat ad curvaturam lineae (o) apud punctum (x, y): vergit insuper (73) angulus $(\theta''x)$ ad angulum (τx); exprimetur igitur curvatura lineae (o) apud punctum (x, y) per

$$lim. \frac{1}{\rho} = lim. \frac{\Delta(\theta''x)}{\Delta\chi} = \frac{d(\tau x)}{ds};$$

et facto $lim. \frac{1}{\rho} = \frac{1}{r}$,

$$r = \frac{ds}{d(\tau x)} \ldots (o^v).$$

Circulus radio r descriptus dicitur *osculator*; habet in puncto (x, y) tangentem communem cum (o), centrum vero situm alicubi in respondente normali: ipse praeterea r vocatur *radius osculi*, vel etiam *curvaturae* seu *curvedinis*.

85. Sumpta x pro variabili independente, erunt (73 : 11 : 77)

$$d(\tau x) = d\, arc(tang = f'(x)) = \frac{f''(x)dx}{1 + f'^2(x)},$$

$$ds = dx\sqrt{1 + f'^2(x)}.$$

Ad haec: singulis punctis curvae (o) unicus est radius osculi parti cavae semper obversus; et quia ipsa cavitas curvae vel obverti potest abscissarum axi, vel

oppositam respicere plagam, hinc si ad habendam rationem istiusmodi positionum radius ille in primo casu censetur positivus, censendus erit negativus in secundo. Non pluribus opus est ut intelligamus formulam (o^{V}) verti (80) in

$$r = -\frac{[1+f'^2(x)]^{\frac{3}{2}}}{f''(x)}, \text{ seu } (73.\ o')\ r = -\frac{(\frac{\nu}{y})^3}{f''(x)} \dots (o^{VI}).$$

Exempla.

I.° Aequatio (o^{IV}. 73. I.°) praebet

$$f'^2(x) = \frac{2a}{y} - 1,$$

ex qua differentiata emergit

$$f'(x)f''(x)dx = -\frac{ady}{y^2},$$

$$f'(x)f''(x) = -\frac{a}{y^2} \cdot \frac{dy}{dx} = -\frac{a}{y^2}f'(x);$$

ideoque (73. I.°)

$$f''(x) = -\frac{a}{y^2} = -\frac{2ay}{2y^3} = -\frac{\nu^2}{2y^3}:$$

et consequenter in cycloide (o^{IV})

$$r = 2\nu.$$

II.° Aequatio (196. i_{27} ex p. 2.^{a})

$$A'x^2 + B'y^2 + 2D'x = K'$$

ad lineas secundi ordinis suppeditat

$$A'x + B'yf'(x) + D' = 0,$$

unde

$$B'^2 y^2 f'^2(x) = A'^2 x^2 + 2A'D'x + D'^2 =$$

$$A'(A'x^2 + 2D'x) + D'^2 = A'(K' - B'y^2) + D'^2 ;$$

ideoque

$$f'^2(x) = \frac{A'K' + D'^2}{B'^2 y^2} - \frac{A'}{B'} .$$

Haec differentiata praebebit

$$f'(x) f''(x) dx = -\frac{A'K' + D'^2}{B'^2} \cdot \frac{dy}{y^3} ,$$

$$f''(x) = -\frac{A'K' + D'^2}{B'^2} \cdot \frac{1}{y^3} :$$

est autem (196 : 205. 4.° : 206. 4.° ex p. 2.^{a})

$$\frac{A'K' + D'^2}{B'^2} = p^2 ,$$

aequalis nimirum quadrato semiparametri ; igitur

$$f''(x) = -\frac{p^2}{y^3} , \text{ et consequenter } r = \frac{y^3}{p^2} .$$

86. Determinandae nunc sint coordinatae v, u illius puncti, in quo situm est centrum circuli osculatoris. Per coordinatas v, u explebuntur simul binae aequationes (73. o$''$: 84 : item 172. II.° ex p. 2.^{a})

$$u - y = -\frac{1}{f'(x)}(v - x) , \ (u - y)^2 + (v - x)^2 = r^2 \ldots (o^{VII}) :$$

propterea

$$(u - y)^2 = \frac{r^2}{1 + f'^2(x)} = \frac{[1 + f'^2(x)]^2}{f''^2(x)} ;$$

et quoniam $u <$ vel $> y$ prout curva (o) cavitatem vel convexitatem obvertit abscissarum axi, ideo (80)

$$\left.\begin{array}{l} u-y=\dfrac{1+f'^2(x)}{f''(x)}; \\ \text{ac proinde ob primam } (o^{VII}) \\ v-x=-\dfrac{1+f'^2(x)}{f''(x)}f'(x). \end{array}\right\} (o^{VIII})$$

87. Eliminatis x, y ex (o) et (o^{VIII}), prodibit aequatio

$$u=\varphi(v) \ldots (o^{IX})$$

ad lineam, ubi reperiuntur centra circulorum omnium qui curvam (o) osculantur.

Exempla.

I.° Quoad cycloidem (o^{IV}. 73. I.°) habemus (85. I.°)

$$u-y=-2y, \text{ unde } u=-y:$$

signum negativum respicit ad positionem ordinatae u; quare consideratis dumtaxat valoribus absolutis,

$$u=y, \; du=dy:$$

rursus (85. I.°)

$$v-x=2y\left(\frac{2a}{y}-1\right)^{\frac{1}{2}}, \text{ unde } v=x+2y\left(\frac{2a}{y}-1\right)^{\frac{1}{2}};$$

et consequenter (73. I.° o^{IV})

$$dv=dx+2\left(\frac{2a}{y}-1\right)^{\frac{1}{2}}dy+2yd\left(\frac{2a}{y}-1\right)^{\frac{1}{2}}=$$

$$\frac{dy}{\left(\frac{2a}{y}-1\right)^{\frac{1}{2}}}+2\left(\frac{2a}{y}-1\right)^{\frac{1}{2}}dy-\frac{2a}{y}\cdot\frac{dy}{\left(\frac{2a}{y}-1\right)^{\frac{1}{2}}}=$$

$$-\frac{(\frac{2a}{y}-1)dy}{(\frac{2a}{y}-1)^{\frac{1}{2}}}+2(\frac{2a}{y}-1)^{\frac{1}{2}}dy=(\frac{2a}{y}-1)^{\frac{1}{2}}dy,$$

seu

$$dv=(\frac{2a}{u}-1)^{\frac{1}{2}}du=du\sqrt{[\frac{2a-u}{u}]}.$$

Inferimus (73. I.° o‴) centra circulorum, qui osculantur cycloidem, fore omnia in altera cycloide genita eodem circulo ac illa altera.

II.° In lineis secundi ordinis (85. II.°)

$$f''(x)=-\frac{p^2}{y^3};$$

praeterea $f'(x)=\frac{p}{y}$ in parabola

$$y^2=2px,$$

$f'(x)=-\frac{b^2x}{a^2y}$ in ellipsi

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

$f'(x)=\frac{b^2x}{a^2y}$ in hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1.$$

Igitur 1.° quoad parabolam assequimur

$$a-y=-\frac{y^3+p^2y}{p^2},\ v-x=\frac{y^2+p^2}{p}:$$

ex istarum prima

$$p^2u = -y^3 \,,\; u^2 = \frac{y^6}{p^4} = \frac{8x^3}{p} \,;$$

e secunda

$$v = 3x + p \,,\; x = \frac{v-p}{3} \,;$$

quare

$$u^2 = \frac{8}{27p}(v-p)^3.$$

2.° quoad ellipsim

$$u - y = \frac{b^2}{a^2} \cdot \frac{y^3}{p^2} - \frac{y^3}{p^2} - y = \frac{y^3}{b^3}\left(b - \frac{a^2}{b}\right) - y \,,$$

$$v - x = \left[\frac{y^3}{b^3}\left(b - \frac{a^2}{b}\right) - y\right]\frac{b^2x}{a^2y} =$$

$$\left[\frac{y^2}{b^2}\left(1 - \frac{a^2}{b^2}\right) - 1\right]\frac{b^2x}{a^2} = \frac{x^3}{a^3}\left(a - \frac{b^2}{a}\right) - x \,:$$

hinc, factis compendii causa

$$b - \frac{a^2}{b} = h \,,\; a - \frac{b^2}{a} = k \,,$$

prodeunt

$$\frac{u}{h} = \frac{y^3}{b^3} \,,\; \frac{v}{k} = \frac{x^3}{a^3} \,,$$

seu

$$\left(\frac{u}{h}\right)^{\frac{2}{3}} = \frac{y^2}{b^2} \,,\; \left(\frac{v}{k}\right)^{\frac{2}{3}} = \frac{x^2}{a^2} \;;$$

ideoque

$$\left(\frac{v}{k}\right)^{\frac{2}{3}}+\left(\frac{u}{h}\right)^{\frac{2}{3}}=1.$$

3.° simili modo quoad hyperbolam.

$$\left(\frac{v}{k}\right)^{\frac{2}{3}}-\left(\frac{u}{h}\right)^{\frac{2}{3}}=1\,;$$

ubi

$$k=a+\frac{b^2}{a},\ h=b+\frac{a^2}{b}.$$

88. Radii circulorum curvam (o) osculantium sunt totidem tangentes respondentis curvae (o^{IX}. 87).

Differentiatis enim (o^{VIII}. 86), provenient

$$du-f'(x)dx=d\frac{1+f'^2(x)}{f''(x)},$$

$$dv-dx=-f''(x)\frac{1+f'^2(x)}{f''(x)}dx-f'(x)d\frac{1+f'^2(x)}{f''(x)}\,;$$

et eliminato $d\dfrac{1+f'^2(x)}{f''(x)}$,

$$du=-\frac{dv}{f'(x)}\ \text{seu}\ \frac{du}{dv}\cdot\frac{dy}{dx}+1=0.$$

Jam non pluribus opus est (73 : 84 : item 172. I.° 4.° ex p. 2.a) ut pateat veritas assertionis.

89. Binarum (o^{VII}. 86) prima vertitur (88) in

$$u-y=\frac{du}{dv}(v-x),$$

secunda praebet

$$(u-y)(du-dy)+(v-x)(dv-dx)=rdr\ ;$$

igitur (88)

$$v-x=\frac{rdrdv}{dv^2+du^2-dvdx-dudy}=\frac{rdrdv}{dv^2+du^2},$$

$$u-y=\frac{rdrdu}{dv^2+du^2-dvdx-dudy}=\frac{rdrdu}{dv^2+du^2}:$$

et adhibitis substitutionibus in secunda (o^{VII}, 86),

$$\frac{dr^2}{dv^2+du^2}=1,\ dr=\sqrt{dv^2+du^2}.$$

Differentiale nimirum radii osculi ad curvam (o) spectantis aequat (77) differentiale respondentis arcus in curva (o^{IX}, 87); ideoque (8) ipse radius aut aequalis est arcui respondenti in eadem (o^{IX}), aut differt per quantitatem constantem.

90. Pronum est inde colligere illud: si curvae (o^{IX}) advolvatur filum, tum evolvatur ita ut ejus pars libera et maneat distenta, et curvam (o^{IX}) perpetuo tangat (88), extremum fili caput, quod in evolutionis initio ponimus in (o), manebit semper in eadem (o), ipsamque describet. Duarum (o^{IX}), (o) prior dicitur *evoluta*, posterior genita evolutione illius.

91. Aliquid subjungimus de radio osculi in curvis, quae ad coordinatas polares (75) z, ω referuntur.

In prima (o^{VI}, 85) desinat x esse variabilis independens, ut pro tali habeatur arcus ω: mutanda erit (15) $\frac{d^2y}{dx^2}$ in $\frac{dxd^2y-dyd^2x}{dx^3}$, unde

$$r=-\frac{dx^3\left(1+\frac{dy^2}{dx^2}\right)^{\frac{3}{2}}}{dxd^2y-dyd^2x}=-\frac{(dx^2+dy^2)^{\frac{3}{2}}}{dxd^2y-dyd^2x}.$$

Jam vero (75)

$$dx^2 + dy^2 = dz^2 + z^2 d\omega^2,$$

$$dxd^2y - dyd^2x = 2dz^2 d\omega - zd\omega d^2 z + z^2 d\omega^3;$$

igitur

$$r = -\frac{(dz^2 + z^2 d\omega^2)^{\frac{3}{2}}}{2dz^2 d\omega - zd\omega d^2 z + z^2 d\omega^3}.$$

Simili modo (15 : 86. o^{VIII})

$$u-y = \frac{(dx^2+dy^2)dx}{dxd^2y-dyd^2x} = \frac{(dz^2+z^2d\omega^2)(\cos\omega dz - z\sin\omega d\omega)}{2dz^2d\omega - zd\omega d^2z + z^2d\omega^3},$$

$$v-x = -\frac{(dx^2+dy^2)dy}{dxd^2y-dyd^2x} = -\frac{(dz^2+z^2d\omega^2)(\sin\omega dz + z\cos\omega d\omega)}{2dz^2d\omega - zd\omega d^2z + z^2d\omega^3}.$$

Pone $\omega = \frac{\pi}{2}$; quod eo redit ut x computetur a polo in recta ad perpendiculum insistente radio vectori, ideoque y in ipso radio vectore: habebis $x=0$, $y=z$, $\sin\omega = 1$, $\cos\omega = 0$, ac proinde

$$u = z - \frac{z(dz^2 + z^2 d\omega^2)}{2dz^2 - zd^2z + z^2 d\omega^2},$$

$$v = -\frac{(dz^2 + z^2 d\omega^2)dz}{2dz^2 d\omega - zd\omega d^2 z + z^2 d\omega^3}.$$

Exemplum.

Sit curva

$$z = e^{\frac{\omega}{a}},$$

quam dicunt *spiralem logarithmicam*: patet ejusmo-

di curvam infinitis spiris circa polum ita converti, ut radius vector cum tangente contineat angulum $(z\tau)$ semper eumdem; siquidem (121 : 123. i' ex p. 2.ª)

$$tang(z\tau) = tang[90° - (\tau x)] =$$

$$cot(\tau x) = \frac{dx}{dy} = \frac{cos\omega dz - zsin\omega d\omega}{sin\omega dz + zcos\omega d\omega} = -\frac{zd\omega}{dz} = -a;$$

ad haec: ex $d\omega = \frac{adz}{z}$ assequimur $0 = \frac{azd^2z - adz^2}{z^2}$,

$d^2z = \frac{dz^2}{z}$, unde

$$r = -\frac{z}{a}(1 + a^2)^{\frac{1}{2}},\ u = 0,\ v = -\frac{z}{a}.$$

Jam istarum aequationum prima ostendit radium osculi esse radio vectori proportionalem; caeterae demonstrant et centrum circuli osculatoris reperiri constanter in linea recta transeunte per polum radioque vectori z ad perpendiculum insistente, et distantiam ejusdem centri ab ipso polo fore $v = -\frac{z}{a}$. Hinc radius vector v curvae evolutae cum radio osculi r idest (88) cum recta tangente continebit angulum, cujus tangens trigonometrica (125 ex p. 2.ª)

$$tang(vr) = \frac{z}{v} = -a.$$

Inferimus curvam evolutam nihil esse aliud nisi spiralem logarithmicam aequalem genitae.

92. Quod tandem spectat ad curvas osculatrices, haec notamus. 1.° binae curvae (o) et $y = f(x)$ ibi dicuntur sese osculari ubi et tangentem, et circulum osculatorem habent communem, obvertuntque cavitatem ad eamdem partem: igitur quoad osculationis pun-

etum valebunt simul (73 : 80 : 85)

$$f(x) = \mathrm{f}(x)\,,\ f'(x) = \mathrm{f}'(x)\,,\ f''(x) = \mathrm{f}''(x).$$

2.° sit λ nova quantitas variabilis, cujus relatio ad x et y exhibeatur per

$$\lambda = \varphi(x, y):$$

differentialia semel iterumque capientes in hypothesi λ independentis assequemur (45 : 46)

$$\left.\begin{aligned} 1 &= \frac{d\varphi}{dx} \cdot \frac{dx}{d\lambda} + \frac{d\varphi}{dy} \cdot \frac{dy}{d\lambda}\,, \\ 0 &= \frac{d\varphi}{dx} \cdot \frac{d^2x}{d\lambda^2} + \frac{d^2\varphi}{dx^2} \cdot \frac{dx^2}{d\lambda^2} + \\ &\frac{d\varphi}{dy} \cdot \frac{d^2y}{d\lambda^2} + \frac{d^2\varphi}{dy^2} \cdot \frac{dy^2}{d\lambda^2} + 2\,\frac{d^2\varphi}{dxdy} \cdot \frac{dx}{d\lambda} \cdot \frac{dy}{d\lambda}\,; \end{aligned}\right\} (o^{\mathrm{x}})$$

designatis insuper tam $f'(x)$, $f''(x)$ quam $\mathrm{f}'(x)$, $\mathrm{f}''(x)$ per y', y'', erunt (14 : 15)

$$y' = \frac{\dfrac{dy}{d\lambda}}{\dfrac{dx}{d\lambda}}\,,\ y'' = \frac{\dfrac{dx}{d\lambda} \cdot \dfrac{d^2y}{d\lambda^2} - \dfrac{dy}{d\lambda} \cdot \dfrac{d^2x}{d\lambda^2}}{\dfrac{dx^3}{d\lambda^3}} \ldots (o^{\mathrm{xI}}).$$

Ex (o^{x}) et (o^{xI}) profluunt

$$\frac{dx}{d\lambda}\,,\ \frac{dy}{d\lambda}\,,\ \frac{d^2x}{d\lambda^2}\,,\ \frac{d^2y}{d\lambda^2} \ldots (o^{\mathrm{xII}}).$$

expressae per

$$y'\,,\ y''\,,\ \frac{d\varphi}{dx}\,,\ \frac{d\varphi}{dy}\,,\ \frac{d^2\varphi}{dx^2}\,,\ \frac{d^2\varphi}{dy^2}\,,\ \frac{d^2\varphi}{dxdy}:$$

sed hae quantitates quoad osculationis punctum retinent (1.°) eumdem valorem quum ab una curva transitur ad alteram; idipsum ergo dicendum de quanti-

tatibus (o^{XII}). 3.° mutua curvarum (o) et $y=\mathrm{f}(x)$ appropinquatio in viciniis puncti (x, y) desumi poterit ex valore infinitesimae quantitatis

$$f(x+\beta)-\mathrm{f}(x+\beta) \ldots (o^{XIII}) :$$

certe si ordinem infinitesimae (o^{XIII}) exprimit numerus c, curvarum (o) et $y=\mathrm{f}(x)$ altera ad alteram in viciniis puncti (x, y) accedet maxime (251 ex p. 1.a) aliarum omnium, quibus respondet ordo ipsius (o^{XIII}) expressus per numerum $< c$. 4.° derivatas ex (o^{XIII}) quoad β capientes habemus

$$f'(x+\beta)-\mathrm{f}'(x+\beta),\ f''(x+\beta)-\mathrm{f}''(x+\beta),\ \ldots$$
$$f'''(x+\beta)-\mathrm{f}'''(x+\beta), \ldots\ldots$$

Sed, evanescente β, prima, quae inter ejusmodi derivatas non evanescit, est ordinis (29) vel $=c$, vel immediate $> c$: ergo

$$\left.\begin{array}{l} f(x)=\mathrm{f}(x),\ f'(x)=\mathrm{f}'(x),\ f''(x)=\mathrm{f}''(x), \\ \quad f'''(x)=\mathrm{f}'''(x), \ldots\ldots \end{array}\right\} (o^{XIV})$$

usque ad ordinem vel $=c$, vel immediate $< c$; eumdem scilicet in utraque curva retinebunt valorem, non solum (1.°) ordinatae puncti (x, y) et quae ab iis promanant derivatae primi ac secundi ordinis, at etiam caeterae usque ad ordinem vel $=c$ vel immediate $< c$. 5.° si e duabus curvis, quae sese debent osculari, altera non est omnino determinata, ut in ejus aequationem v. gr. $y=\mathrm{f}(x)$ ingrediantur constantes arbitrariae a, a', a'', ..., hac poterunt definiri per totidem aequationes (o^{XIV}). 6.° quantitates infinitesimae

$$f(x+\beta)-\mathrm{f}(x+\beta),\ f(x-\beta)-\mathrm{f}(x-\beta)$$

aut iisdem gaudent signis, aut contrariis: in primo casu manifeste se mutuo secabunt osculatrices curvae, haud se secabunt in secundo.

De tangentibus, de normalibus, de plano osculante, deque asymptotis.

93. Imaginemur in spatio curvam ad axes orthogonales AX, AY, AZ relatam; sintque

$$\{ x = f(z) ,\ y = \mathrm{f}(z) \} \ldots (a)$$

ejus projectiones (182 ex p. 2.[a]) in planis XAZ, YAZ: porro sive curva sit plana sive duplici gaudeat curvedine, ipsius tangentem in puncto (x, y, z) determinabunt eae projectionum tangentes, quae in duobus planis coordinatis v. gr. XAZ, YAZ ducuntur per puncta (x, z), (y, z). Itaque designantibus v, u, t coordinatas rectae tangentis curvam (a), erunt (73. o'')

$$\left\{ v - x = \frac{dx}{dz}(t-z) ;\ u - y = \frac{dy}{dz}(t-z) \right\} \ldots (a').$$

Hinc 1.° si per (τx), (τy), (τz) exhibentur anguli, quos recta tangens in puncto (x, y, z) curvam (a) efficit cum axibus AX, AY, AZ, facto compendii causa

$$\sqrt{\left[1 + \left(\frac{dx}{dz}\right)^2 + \left(\frac{dy}{dz}\right)^2\right]} = h,$$

erunt (183 : 184 ex p. 2.[a])

$$\cos(\tau x) = \frac{1}{h}\frac{dx}{dz},\ \cos(\tau y) = \frac{1}{h}\frac{dy}{dz},\ \cos(\tau z) = \frac{1}{h} \ldots (a'').$$

2.° si duae curvae (a) et

$$\{ x = \varphi(z) ,\ y = \psi(z) \} \ (b)$$

se mutuo tangunt (73), erunt quoad contactus punctum

$$f(z)=\varphi(z)\ ,\ \ \mathrm{f}(z)=\psi(z)\ ,\ \ f'(z)=\varphi'(z)\ ,\ \ \mathrm{f}'(z)=\psi'(z).$$

94. Facile intelligitur, etsi curva (a) non est plana, adhuc tamen locum fore theorematî jam demonstrato (76). Hoc posito, contemplemur in (a) arcum s una cum incremento Δs : si c denotat chordam qua subtenditur Δs, erit (176 ex p. 2.ª)

$$c=\sqrt{\Delta x^2+\Delta y^2+\Delta z^2}\ ,\ \ \frac{\Delta s}{c}=\frac{\Delta s}{\Delta z\sqrt{\left[1+\frac{\Delta x^2}{\Delta z^2}+\frac{\Delta y^2}{\Delta z^2}\right]}}\ ,$$

unde

$$lim.\frac{\Delta s}{c}=1=\frac{1}{h}\frac{ds}{dz}\ ,$$

et consequenter $ds=hdz=\sqrt{dx^2+dy^2+dz^2}\ \ldots\ (a''')$.

Patet nunc formulas (a'') sic scribi posse

$$cos(\tau x)=\frac{dx}{ds}\ ,\ \ cos(\tau y)=\frac{dy}{ds}\ ,\ \ cos(\tau z)=\frac{dz}{ds}\ \ldots\ (a^{\mathrm{IV}})\ ;$$

ex quibus emergit (186 ex p. 2.ª) aequatio

$$(v-x)dx+(u-y)dy+(t-z)dz=0\ \ldots\ (a^{\mathrm{V}})$$

inter coordinatas v, u, t plani transeuntis per punctum (x, y, z) et ad perpendiculum insistentis rectae tangenti ibidem curvam (a) : in plano (a^{V}) jacent normales omnes (sunt numero infinitae), quae per contactus punctum duci possunt ad ipsam (a) utcumque sitam in spatio.

95. Denotet σ infinitesimam longitudinem computatam a contactu versus eamdem plagam et in curva (a), et in tangente (a') : extrema puncta longitudinis σ, alterum in (a), alterum in (a'), jungat recta linea δ ; ipsisque respondeant coordinatae x_1, y_1, z_1 in (a)

et v_1, u_1, t_1 in (a'): habeatur insuper s pro variabili independente, ut in hac hypothesi investigentur cosinus angulorum (δx), (δy), (δz), quos recta δ efficit cum axibus AX, AY, AZ.

Coordinatae v_1, u_1, t_1 exhiberi possunt per hunc modum (94. a^{IV})

$$v_1 = x + \sigma cos(\tau x) = x + \sigma \frac{dx}{ds},$$

$$u_1 = y + \sigma cos(\tau y) = y + \sigma \frac{dy}{ds},$$

$$t_1 = z + \sigma cos(\tau z) = z + \sigma \frac{dz}{ds}.$$

Coordinatae x_1, y_1, z_1 sic possunt exprimi (23, y^{VIII})

$$x_1 = x + \sigma \frac{dx}{ds} + \frac{\sigma^2}{2}\left(\frac{d^2 x}{ds^2} + \alpha'\right),$$

$$y_1 = y + \sigma \frac{dy}{ds} + \frac{\sigma^2}{2}\left(\frac{d^2 y}{ds^2} + \alpha''\right),$$

$$z_1 = z + \sigma \frac{dz}{ds} + \frac{\sigma^2}{2}\left(\frac{d^2 z}{ds^2} + \alpha'''\right);$$

vergunt α', α'', α''' una cum σ ad *lim.* $= 0$. Igitur (125 ex p. 2.ª)

$$\left.\begin{aligned} cos(\delta x) &= \frac{x_1 - v_1}{\delta} = \frac{\sigma^2}{2\delta}\left(\frac{d^2 x}{ds^2} + \alpha'\right), \\ cos(dy) &= \frac{y_1 - u_1}{\delta} = \frac{\sigma^2}{2\delta}\left(\frac{d^2 y}{ds^2} + \alpha''\right), \\ cos(\delta z) &= \frac{z_1 - t_1}{\delta} = \frac{\sigma^2}{2\delta}\left(\frac{d^2 z}{ds^2} + \alpha'''\right). \end{aligned}\right\} (a^{VI})$$

Ad δ quod spectat, cum habeamus (176 ex p. 2.^{a})

$$\delta = [(x_1 - v_1)^2 + (y_1 - u_1)^2 + (z_1 - t_1)^2]^{\frac{1}{2}},$$

erit

$$\delta = \frac{\sigma^2}{2}[(\frac{d^2x}{ds^2} + \alpha')^2 + (\frac{d^2y}{ds^2} + \alpha'')^2 + (\frac{d^2z}{ds^2} + \alpha''')^2]^{\frac{1}{2}} \ldots (a^{VII})$$

96. Nunc facile invenitur cosinus anguli $(\delta\tau)$, quem recta δ efficit cum tangente: nam (177 ex p. 2.^{a})

$$cos(\delta\tau) = cos(\delta x)cos(\tau x) + cos(\delta y)cos(\tau y) + cos(\delta z)cos(\tau z);$$

et consequenter (95 : 94. a^{IV})

$$cos(\delta\tau) =$$

$$\frac{\frac{dx}{ds}(\frac{d^2x}{ds^2} + \alpha') + \frac{dy}{ds}(\frac{d^2y}{ds^2} + \alpha'') + \frac{dz}{ds}(\frac{d^2z}{ds^2} + \alpha''')}{[(\frac{d^2x}{ds^2} + \alpha')^2 + (\frac{d^2y}{ds^2} + \alpha'')^2 + (\frac{d^2z}{ds^2} + \alpha''')^2]^{\frac{1}{2}}}.$$

Numerator istius fractionis reducitur ad

$$\alpha'\frac{dx}{ds} + \alpha''\frac{dy}{ds} + \alpha'''\frac{dz}{ds};$$

siquidem (94. α''')

$$dxd^2x + dyd^2y + dzd^2z = dsd^2s = 0:$$

Inferimus, vergente σ ad *lim.* $= 0$, angulum $(\delta\tau)$ accessurum (95) usque adeo ad $90°$, donec, facta $\delta = 0$ in contactus puncto, evadat $(\delta\tau) = 90°$.

97. Itaque directio lineae rectae σ vergit ad directionem cujusdam e normalibus numero (94) infinitis quae duci possunt per punctum (x, y, z) curvae (a). Peculiaris ista normalis (dicitur *principalis*) exhibeatur per v_1; erunt (95. a^{VI}. a^{VII})

$$\left.\begin{aligned} \cos(\nu_1,x) &= \frac{d^2x}{[(d^2x)^2+(d^2y)^2+(d^2z)^2]^{\frac{1}{2}}}, \\ \cos(\nu_1,y) &= \frac{d^2y}{[(d^2x)^2+(d^2y)^2+(d^2z)^2]^{\frac{1}{2}}}, \\ \cos(\nu_1,z) &= \frac{d^2z}{[(d^2x)^2+(d^2y)^2+(d^2z)^2]^{\frac{1}{2}}}: \end{aligned}\right\} (a^{\mathrm{VIII}})$$

hinc vero (186 ex p. 2.^{a}) aequationes

$$\frac{v-x}{d^2x} = \frac{u-y}{d^2y} = \frac{t-z}{d^2z} \ldots (a^{\mathrm{IX}})$$

ad normalem ν_1; necnon (ibid.) aequatio

$$(v-x)d^2x + (u-y)d^2y + (t-z)d^2z = 0 \ldots (a^{\mathrm{X}})$$

ad planum, quod et transit per punctum (x, y, z), et perpendiculariter insistit ipsi ν_1.

98. Planum ductum per tangentem et normalem principalem dicitur *osculans*. Finge tibi (73. 6.^{o} ex p. 2.^{a}) rectam k transeuntem per contactus punctum (x, y, z) ac perpendiculariter insistentem plano osculanti: quoniam (69 ex p. 2.^{a})

$$(k\tau) = 90^\circ, \ (k\nu_1) = 90^\circ,$$

ideo (177 ex p. 2.^{a})

$$\cos(k\tau) = \cos(kx)\cos(\tau x) + \cos(ky)\cos(\tau y) + \cos(kz)\cos(\tau z) = 0, \ \cos(k\nu_1) = \cos(kx)\cos(\nu_1 x) + \cos(ky)\cos(\nu_1 y) + \cos(kz)\cos(\nu_1 z) = 0;$$

et consequenter (94. a^{IV} : 97. a^{VIII})

$$\left.\begin{aligned} \cos(kx)dx + \cos(ky)dy + \cos(kz)dz &= 0, \\ \cos(kx)d^2x + \cos(ky)d^2y + \cos(kz)d^2z &= 0. \end{aligned}\right\} (a^{\mathrm{XI}})$$

Binis (a^{XI}) substitui potest (127 ex p. 1.^{a}) formula

$$\left.\begin{array}{l}\dfrac{cos(kx)}{dyd^2z-dzd^2y}=\dfrac{cos(ky)}{dzd^2x-dxd^2z}=\dfrac{cos(kz)}{dxd^2y-dyd^2x}, \\ \text{cui conjungenda (177 ex p. 2.}^a) \\ \qquad cos^2(kx)+cos^2(ky)+cos^2(kz)=1,\end{array}\right\}(a^{\mathrm{XII}})$$

ut inde eruantur $cos(kx)$, $cos(ky)$, $cos(kz)$, sicque determinetur plani osculantis positio. Ad istiusmodi plani aequationem quod pertinet, ea erit (186 ex p. 2.^{a})

$$\left.\begin{array}{l}(v-x)cos(kx)+(u-y)cos(ky)+(t-z)cos(kz)=0, \\ \text{seu ob primam } (a^{\mathrm{XII}}) \\ (v-x)(dyd^2z-dzd^2y)+(u-y)(dzd^2x-dxd^2z)+ \\ \qquad (t-z)(dxd^2y-dyd^2x)=0.\end{array}\right\}(a^{\mathrm{XIII}})$$

Si curva (a) est plana, quisque videt planum (a^{XIII}) nihil fore aliud nisi planum ipsius curvae.

99. Curva in spatio constituta gaudeat asymptotis: istarum projectiones in planis coordinatis erunt asymptoti projectionum illius curvae in iisdem planis. Ratio igitur investigandi asymptotos lineae curvae in spatio utcumque sitae traduci poterit ad methodum (74) determinandi asymptotos linearum curvarum, quae in superficie plana sunt constitutae.

De circulo osculatore; ubi et aliquid generatim annotatur circa evolutas, et circa curvas osculatrices.

100. Puncta (x, y, z), $(x+\Delta x, y+\Delta y, z+\Delta z)$ praeter arcum Δs intercipiant etiam circularem arcum $\Delta\chi$, cujus radius $=\rho$; in punctis illis tangatur $\Delta\chi$ a rectis lineis θ', θ'', et Δs a rectis lineis τ, τ'; exprimat insuper k' rectam perpendiculariter insistentem plano osculanti (98) ducto per $(x+\Delta x, y+\Delta y, z+\Delta z)$. Hic quoque (84) erit (76 : 94)

$$lim. \frac{\Delta\chi}{\Delta s} = 1 ;$$

ac proinde, accedente puncto $(x+\Delta x, y+\Delta y, z+\Delta z)$ ad (x, y, z), mutabitur ρ ita ut curvatura circuli vergat ad curvaturam lineae (a) apud contactus punctum (x, y, z). Porro apud punctum (x, y, z) duplex spectari potest curvedo lineae (a), altera in plano (a^{XIII}), altera in plano (a^{X}) : in utroque casu habetur (83)

$$\frac{1}{\rho} = \frac{(\theta'\theta'')}{\Delta\chi} ;$$

sed quoad primam curvedinem

$$lim. \frac{(\theta'\theta'')}{(\tau\tau')} = 1 ,$$

quoad secundam (98)

$$lim. \frac{(\theta'\theta'')}{(kk')} = 1 :$$

facto igitur $lim. \frac{1}{\rho} = \frac{1}{r}$ in 1.° casu, et $lim. \frac{1}{\rho} = \frac{1}{\mathrm{R}}$ in 2.°, provenient

$$r = lim. \frac{\Delta s}{(\tau\tau')} , \quad \mathrm{R} = lim. \frac{\Delta s}{(kk')} \ldots (a^{\text{XIV}}).$$

E duobus circulis, qui radiis r, R describi possunt, primus dumtaxat dicitur proprie osculator ; habet in puncto (x, y, y) tangentem communem cum (a), centrum vero situm alicubi in respondente (97) $\nu_{\prime}$.

101. Patet illud : si cosinus angulorum, quos altera e duabus τ, τ' efficit cum axibus AX, AY, AZ exprimuntur per

$$cos(\tau x) , \ cos(\tau y) , \ cos(\tau z) ,$$

cosinus angulorum, quos altera continet cum iisdem axibus, poterunt exprimi per

$$cos(\tau x)+\Delta cos(\tau x),\ cos(\tau y)+\Delta cos(\tau y),$$
$$cos(\tau z)+\Delta cos(\tau z).$$

Hinc (177 ex p. 2.^{a})

$$cos(\tau\tau')=cos(\tau x)[cos(\tau x)+\Delta cos(\tau x)]+$$
$$cos(\tau y)[cos(\tau y)+\Delta cos(\tau y)]+cos(\tau z)[cos(\tau z)+\Delta cos(\tau z)],$$
$$1=[cos(\tau x)+\Delta cos(\tau x)]^2+[cos(\tau y)+\Delta cos(\tau y)]^2+$$
$$[cos(\tau z)+\Delta cos(\tau z)]^2,$$
$$1=cos^2(\tau x)+cos^2(\tau y)+cos^2(\tau z);$$

ideoque (127. 3.° ex p. 2.^{a})

$$2[1-cos(\tau\tau')]=[2sin\tfrac{1}{2}(\tau\tau')]^2=$$
$$[\Delta cos(\tau x)]^2+[\Delta cos(\tau y)]^2+[\Delta cos(\tau z)]^2.$$

Exinde transitur ad

$$\frac{\Delta s^2}{[2sin\frac{1}{2}(\tau\tau')]^2}=\frac{\Delta s^2}{(\tau\tau')^2}\left[\frac{\frac{1}{2}(\tau\tau')}{sin\frac{1}{2}(\tau\tau')}\right]^2=$$
$$\frac{1}{\left[\frac{\Delta cos(\tau x)}{\Delta s}\right]^2+\left[\frac{\Delta cos(\tau y)}{\Delta s}\right]^2+\left[\frac{\Delta cos(\tau z)}{\Delta s}\right]^2};$$

et facto prius ad limites gradu (129. 2.° ex p. 2.^{a}), tum extracta radice quadrata, emerget (100. a^{XIV}: 94. a^{IV})

$$r=\frac{1}{\left[\left(\frac{dcos(\tau x)}{ds}\right)^2+\left(\frac{dcos(\tau y)}{ds}\right)^2+\left(\frac{dcos(\tau z)}{ds}\right)^2\right]^{\frac{1}{2}}}=$$

$$\frac{1}{\left[\left(\frac{d\left(\frac{dx}{ds}\right)}{ds}\right)^2+\left(\frac{d\left(\frac{dy}{ds}\right)}{ds}\right)^2+\left(\frac{d\left(\frac{dz}{ds}\right)}{ds}\right)^2\right]^{\frac{1}{2}}}=$$

$$\frac{1}{\left[\left(\frac{d^2x}{ds^2}\right)^2+\left(\frac{d^2y}{ds^2}\right)^2+\left(\frac{d^2z}{ds^2}\right)^2\right]^{\frac{1}{2}}}\ \ldots\ (a^{XV}).$$

Simili modo assequemur

$$R=\frac{1}{\left[\left(\frac{d\cos(kx)}{ds}\right)^2+\left(\frac{d\cos(ky)}{ds}\right)^2+\left(\frac{d\cos(kz)}{ds}\right)^2\right]^{\frac{1}{2}}}\ldots(a^{XVI}).$$

102. Determinandae sint coordinatae v, u, t illius puncti, in quo situm est centrum circuli osculatoris.

Quoniam centrum istud invenitur alicubi (100) in ν_1, ideo (125 ex p. 2.^{a})

$$\left.\begin{array}{c}\dfrac{v-x}{r}=\cos(\nu_1 x),\ \dfrac{u-y}{r}=\cos(\nu_1 y),\\[2ex] \dfrac{t-z}{r}=\cos(\nu_1 z)\end{array}\right\}(a^{XVII}),$$

seu (97. a^{VIII} : 101. a^{XV})

$$\frac{v-x}{r}=r\frac{d^2x}{ds^2},\ \frac{u-y}{r}=r\frac{d^2y}{ds^2},\ \frac{t-z}{r}=r\frac{d^2z}{ds^2};$$

ac proinde

$$v-x=r^2\frac{d^2x}{ds^2},\ u-y=r^2\frac{d^2y}{ds^2},\ t-z=r^2\frac{d^2z}{ds^2}\ldots(a^{XVIII}).$$

103. Si desinit s repraesentare variabilem independentem ut pro tali habeatur v. gr. z, adhibendae erunt (15)

$$\frac{dsd^2x-dxd^2s}{ds^3},\ \frac{dsd^2y-dyd^2s}{ds^3},\ -\frac{dzd^2s}{ds^3}$$

pro

$$\frac{d^2x}{ds^2},\ \frac{d^2y}{ds^2},\ \frac{d^2z}{ds^2};$$

sic (a^{xv}) vertetur (94. a''') in

$$r=\frac{[1+f'^2(z)+\mathrm{f}'^2(z)]^{\frac{3}{2}}}{[(f'(z)\mathrm{f}''(z)-\mathrm{f}'(z)f''(z))^2+f''^2(z)+\mathrm{f}''^2(z)]^{\frac{1}{2}}}\ldots(a^{xix}),$$

et (a^{xviii}) in

$$\left.\begin{aligned}
\nu-x&=\frac{\mathrm{f}'(z)[\mathrm{f}'(z)f''(z)-f'(z)\mathrm{f}''(z)]+f''(z)}{[1+f'^2(z)+\mathrm{f}'^2(z)]^2}r^2,\\
u-y&=\frac{f'(z)[f'(z)\mathrm{f}''(z)-\mathrm{f}'(z)f''(z)]+\mathrm{f}''(z)}{[1+f'^2(z)-\mathrm{f}'^2(z)]^2}r^2,\\
t-z&=-\frac{f'(z)f''(z)+\mathrm{f}'(z)\mathrm{f}''(z)}{[1+f'^2(z)+\mathrm{f}'^2(z)]^2}r^2.
\end{aligned}\right\}(a^{xx})$$

104. Eliminatis x et y ex (a) et (a^{xviii}), prodibunt tres aequationes

$$\nu-f(z)=f_1(z),\ u-\mathrm{f}(z)=f_2(z),\ t-z=f_3(z);$$

e quibus eliminata z, emergent binae

$$\left\{\nu=f_4(t),\ u=f_5(t)\right\}(a^{xxi})$$

spectantes ad lineam, ubi reperiuntur centra circulorum omnium qui curvam (a) osculantur.

105. Formulae (a^{xvii}) praebent (177 ex p. 2.^{a})

$$(\nu-x)^2+(u-y)^2+(t-z)^2=r^2\ \ldots\ (a^{xxii});$$

formulae (a^{xviii}) multiplicatae, prima per d^2x, secunda per d^2y, tertia per d^2z, suppeditant (101. a^{xv})

$$(v-x)d^2x+(u-y)d^2y+(t-z)d^2z=ds^2,$$

seu (94. a''')

$$\left.\begin{array}{c}(v-x)d^2x+(u-y)d^2y+(t-z)d^2z-\\ dx^2-dy^2-dz^2=0\end{array}\right\}(a^{XXIII}):$$

quisque videt aequationes (a^{XXII}), (a^{XXIII}) et $(a^{V}.\ 94)$ expleri simul per coordinatas v, u, t lineae (a^{XXI}). Ob (a^{XXIII}) traducitur (a^{V}) differentiata ad

$$dvdx+dudy+dtdz=0\ldots(a^{XXIV});$$

hinc si denotetur per ς arcus lineae (a^{XXI}), et per τ_1 recta tangens in puncto (v, u, t) ipsam (a^{XXI}), erit (94. a^{IV}: item 177 ex p. 2.a)

$$cos(\tau\tau_1)=\frac{dv}{d\varsigma}\cdot\frac{dx}{ds}+\frac{du}{d\varsigma}\cdot\frac{dy}{ds}+\frac{dt}{d\varsigma}\cdot\frac{dz}{ds}=0,$$

ac proinde $(\tau\tau_1)=90°$. Ob (a^{V}) traducitur (a^{XXV}) differentiata ad

$$(v-x)dv+(u-y)du+(t-z)dt=rdr,$$

unde (102. a^{XVIII}: 94. a^{IV}: item 177 ex p. 2.a)

$$\left.\frac{dr}{d\varsigma}=\frac{v-x}{r}\cdot\frac{dv}{d\varsigma}+\frac{u-y}{r}\cdot\frac{du}{d\varsigma}+\frac{t-z}{r}\cdot\frac{dt}{d\varsigma}=cos(\nu_1\tau_1).\right\}(a^{XXV})$$

In hypothesi dumtaxat curvae (a) planae et recta ν_1 incidet constanter (88) in rectam τ_1, et valores proinde differentialium dr, $d\varsigma$ exsistent (89) constanter aequales.

106. Curvae (a) advolvatur filum, tum evolvatur ita ut ejus pars libera et maneat distenta, et curvam (a) perpetuo tangat (93); extremum fili caput describet quamdam curvam

$$\{v=F(t),\ u=\mathfrak{f}(t)\}\ (a^{XXVI}):$$

inhaerentes denominationibus jam adhibitis (90) cur-

vam (a) dicimus evolutam, (a^{XXVI}) genitam illius evolutione. Circa istiusmodi curvas haec notamus. 1.° designante λ distantiam inter duo puncta (x, y, z), (v, u, t) sibi mutuo respondentia in (a) et (a^{XXVI}), erunt (94. a^{IV})

$$\frac{v-x}{\lambda} = \cos(\lambda x) = \cos(\tau x) = \frac{dx}{ds},$$

$$\frac{u-y}{\lambda} = \cos(\lambda y) = \cos(\tau y) = \frac{dy}{ds},$$

$$\frac{t-z}{\lambda} = \cos(\lambda z) = \cos(\tau z) = \frac{dz}{ds};$$

unde

$$v-x = \lambda\frac{dx}{ds},\ u-y = \lambda\frac{dy}{ds},\ t-z = \lambda\frac{dz}{ds} \ldots (a^{\text{XXVII}}):$$

est insuper

$$\lambda + s = const.\ \text{ideoque}\ d\lambda = -ds \ldots (a^{\text{XXVIII}}).$$

Aequationes igitur (a^{XXVII}) differentiatae dabunt

$$dv = -\lambda d(\frac{dx}{d\lambda}),\ du = -\lambda d(\frac{dy}{d\lambda}),\ dt = -\lambda d(\frac{dz}{d\lambda}) \ldots (a^{\text{XXIX}}).$$

Praeterea (94. a''')

$d\lambda^2 = ds^2 = dx^2 + dy^2 + dz^2$, et consequenter

$$(\frac{dx}{d\lambda})^2 + (\frac{dy}{d\lambda})^2 + (\frac{dz}{d\lambda})^2 = 1,$$

$$\frac{dx}{d\lambda}d(\frac{dx}{d\lambda}) + \frac{dy}{d\lambda}d(\frac{dy}{d\lambda}) + \frac{dz}{d\lambda}d(\frac{dz}{d\lambda}) = 0:$$

aequationes itaque (a^{XXIX}) multiplicatae, prima per dx, secunda per dy, tertia per dz, suppeditabunt

$$dvdx + dudy + dtdz = 0 \ldots (a^{\text{XXX}}).$$

Ex (a^{xxx}) inferimus (94. a^{iv} : item 177 ex p. 2.^{a}) rectas lineas, quarum altera tangit curvam (a) in puncto (x, y, z), altera curvam (a^{xxvi}) in puncto (v, u, t), fore invicem perpendiculares. 2.° data curva evoluta, cognosci poterunt aequationes (a^{xxvi}) ad curvam genitam illius evolutione: in hunc finem substituentur prius in (a^{xxvii}) valores x, y, z, λ expressi per s; dein eliminabitur s. 3.° vicissim data curva (a^{xxvi}), sic poterunt determinari aequationes (a) ad respondentem evolutam: habemus (176 ex p. 2.^{a})

$$(x-v)^2+(y-u)^2+(z-t)^2=\lambda^2 \ldots (a_1);$$

et quia (a^{xxvii}) et (a^{xxviii}) praebent

$$dx=\frac{x-v}{\lambda}d\lambda, \quad dy=\frac{y-u}{\lambda}d\lambda, \quad dz=\frac{z-t}{\lambda}d\lambda \ldots (a_2),$$

ideo (a_1) differentiata traducetur ad

$$\frac{d\lambda}{\lambda}[(x-v)^2+(y-u)^2+(z-t)^2]-$$

$$(x-v)dv-(y-u)du-(z-t)dt=\lambda d\lambda;$$

quae ob ipsam (a_1) recidet in

$$(x-v)dv+(y-u)du+(z-t)dt=0 \ldots (a_3).$$

Differentiantes (a_3) simulque attendentes ad (a_2) obtinebimus

$$(x-v)d^2v+(y-u)d^2u+(z-t)d^2t+$$

$$\frac{d\lambda}{\lambda}[(x-v)dv+(y-u)du+(z-t)dt]-dv^2-du^2-dt^2=0;$$

quae, denotante ς arcum curvae (a^{xxvi}), ob ipsam (a_3) et ob (a'''. 94) vertetur in

$$(x-v)d^2v+(y-u)d^2u+(z-t)d^2t=d\varsigma^2 \ldots (a_4).$$

Differentiantes (a_4) in hypothesi variabilis ς indepen-

dentis, iterumque attendentes ad (a_2) assequemur

$$(x-v)(d^3v+\frac{d\lambda}{\lambda}d^2v)+(y-u)(d^3u+\frac{d\lambda}{\lambda}d^2u)+$$

$$(z-t)(d^3t+\frac{d\lambda}{\lambda}d^2t)-dvd^2v-dud^2u-dtd^2t=0\,;$$

et quoniam $dvd^2v+dud^2u+dtd^2t=d\varsigma d^2\varsigma=0$, iccirco

$$(x-v)d(\lambda d^2v)+(y-u)d(\lambda d^2u)+(z-t)d(\lambda d^2t)=0..(a_5).$$

Formulae (a_3), (a_5) dant

$$\frac{x-v}{dud(\lambda d^2t)-dtd(\lambda d^2u)}=\frac{y-u}{dtd(\lambda d^2v)-dvd(\lambda d^2t)}=$$

$$\frac{z-t}{dvd(\lambda d^2u)-dud(\lambda d^2v)}\,;$$

et consequenter, factis compendii causa

$$dud(\lambda d^2t)-dtd(\lambda d^2u)=Q',\ dtd(\lambda d^2v)-dvd(\lambda d^2t)=Q'',$$

$$dvd(\lambda d^2u)-dud(\lambda d^2v)=Q''',$$

ob (a_4) exsistet

$$\frac{x-v}{Q'}=\frac{y-u}{Q''}=\frac{z-t}{Q'''}=\frac{d\varsigma^2}{Q'd^2v+Q''d^2u+Q'''d^2t}.$$

Est autem (197 ex p. 1.ª)

$$\frac{(x-v)^2}{Q'^2}=\frac{(y-u)^2}{Q''^2}=\frac{(z-t)^2}{Q'''^2}=$$

$$\frac{(x-v)^2+(y-u)^2+(z-t)^2}{Q'^2+Q''^2+Q'''^2}=\frac{\lambda^2}{Q'^2+Q''^2+Q'''^2}\,;$$

igitur

$$\frac{\lambda^2}{Q'^2+Q''^2+Q'''^2}=\frac{d\varsigma^4}{(Q'd^2v+Q''d^2u+Q'''d^2t)^2}\ \ldots\ (a_6).$$

Jam in (a_6) substitue valores v, u, t expressos per ς; proveniet differentialis aequatio primi ordinis

$$\Phi(\lambda, \varsigma, \frac{d\lambda}{d\varsigma}) = 0 \ldots (a_7)$$

inter variabilem ς, incognitam λ, et derivatam $\frac{d\lambda}{d\varsigma}$: pone (a_7) expleri per

$$\lambda = \psi(\varsigma, C) \ldots (a_8),$$

ubi (53) designat C constantem atque arbitrariam quantitatem; si in (a_1), (a_3), (a_4) substituuntur prius valores v, u, t, λ expressi per ς, ac dein eliminatur ς, exsurgent aequationes (a) ad quaesitam evolutam spectantes: caeterum determinatio functionis ψ pendet a calculo integrali. 4.° quia C est arbitraria, tot erunt ejusmodi evolutae quot diversi valores ipsius C; innumeris videlicet curva (a^{xxvi}) gaudebit evolutis. 5.° aequatio inter x, y, z emergens ab eliminatione (3.°) quantitatis ς e binis (a_3), (a_4) pertinebit ad superficiem amplectentem omnes evolutas curvae (a^{xxvi}): sed de his satis; nunc pauca subjungimus circa curvas osculatrices.

107. Ac 1.° cum binae curvae (a) et $(b.$ 93. 2.°) ibi dicantur sese osculari, ubi et tangentem, et circulum osculatorem habent communem, inferimus quoad osculationis punctum praeter (93. 2.°) aequationes

$$f(z) = \varphi(z),\ \mathrm{f}(z) = \psi(z),\ f'(z) = \varphi'(z),\ \mathrm{f}'(z) = \psi'(z),$$

valituras etiam

$$f''(z) = \varphi''(z),\ \mathrm{f}''(z) = \psi''(z):$$

siquidem ex prima ac tertia $(a^{xx}.$ 103) profluunt

$$f''(z) = \frac{v - x - (t - z)f'(z)}{r^2}[1 + f'^2(z) + \mathrm{f}'^2(z)],$$

$$\varphi''(z) = \frac{v - x - (t - z)\varphi'(z)}{r^2}[1 + \varphi'^2(z) + \psi'^2(z)],$$

e secunda ac tertia

$$f''(z) = \frac{u - y - (t - z)f'(z)}{r^2}[1 + f'^2(z) + f'^2(z)],$$

$$\psi''(z) = \frac{u - y - (t - z)\psi'(z)}{r^2}[1 + \varphi'^2(z) + \psi'^2(z)].$$

2.° inde sequitur (92), si (a), et (b) sese osculantur in puncto (x, y, z), binas quoque projectiones $x = f(z)$, $x = \varphi(z)$ sese osculaturas esse in puncto (x, z), itemque $y = f(z)$, $y = \psi(z)$ in puncto (y, z); et viceversa. 3.° hinc denotantibus c, c' ordines infinitesimarum quantitatum

$$f(z+\beta) - \varphi(z+\beta), \quad f(z+\beta) - \psi(z-\beta),$$

et posito vel $c = c'$, vel c v. gr. $< c'$, erunt (92. 5.°) in utroque casu

$$\left.\begin{array}{l} f(z) = \varphi(z),\ f'(z) = \varphi'(z),\ f''(z) = \varphi''(z), \\ \qquad f'''(z) = \varphi'''(z), \text{ et caet.} \ldots \\ \text{simulque} \\ f(z) = \psi(z),\ f'(z) = \psi'(z),\ f''(z) = \psi''(z), \\ \qquad f'''(z) = \psi'''(z), \text{ et caet.} \ldots \end{array}\right\} (a_1)$$

usque ad ordinem aut $= c$, aut immediate $< c$.

SUPERFICIES CURVAE.

De plano tangente, de respondente normali, deque conis et cylindris qui superficiebus curvis circumscribuntur.

108. Sint $z = f(x, y)$ et $t - z_1 = a(v - x_1) + a'(u - y_1)$, altera (180 ex p. 2.^{a}) ad superficiem cur-

vam, altera (180. I.° 2.° ex p. 2.^{a}) ad planum tangens in puncto $(x_1\,,\,y_1\,,\,z_1)$: prima differentiata suppeditat $dz=f_x'(x\,,\,y)dx+f'_y(x\,,\,y)dy$, secunda pariter differentiata praebet $dt=adv+a'du$; ideoque

$$dz-dt=f_x'(x\,,\,y)dx-adv+f_y'(x\,,\,y)dy-a'du.$$

Fac ut coordinatae $x\,,\,y\,,\,z$, itemque $v\,,\,u\,,\,t$ vergant respective ad $x_1\,,\,y_1\,,\,z_1$; erit in limite

$$0=[f'x_1(x_1\,,\,y_1)-a]dx_1+[f'y_1(x_1\,,\,y_1)-a']dy_1\,,$$

et consequenter

$$f'x_1(x_1\,,\,y_1)-a=0\,,\; f'y_1(x_1\,,\,y_1)-a'=0\,,$$
$$a=f'x_1(x_1\,,\,y_1)\,,\; a'=f'y_1(x_1\,,\,y_1).$$

Detracto itaque indice, proveniet aequatio

$$t-z=\frac{dz}{dx}(v-x)+\frac{dz}{dy}(u-y)\;\ldots\;(h)$$

ad planum tangens superficiem

$$z=f(x\,,\,y)\;\ldots\;(h')$$

in puncto $(x\,,\,y\,,\,z)$.

Notetur illud: binae superficies curvae (h') et $z=\mathrm{f}(x\,,\,y)$ ibi dicuntur se tangere, ubi planum tangens habent commune; propterea quoad ejusmodi contactus punctum valebunt simul

$$f(x\,,\,y)=\mathrm{f}(x\,,\,y)\,,\; f'x(x\,,\,y)=\mathrm{f}'x(x\,,\,y)\,,$$
$$f'y(x\,,\,y)=\mathrm{f}'y(x\,,\,y).$$

109. Ab aequatione (h) profluunt (182. IV.° ex p. 2.^{a}) aequationes

$$v-x=-\frac{dz}{dx}(t-z)\;,\; u-y=-\frac{dz}{dy}(t-z)\;\ldots\;(h'')$$

ad respondentem normalem N: hinc facto compendii causa

$$\sqrt{[1+(\frac{dz}{dx})^2+(\frac{dz}{dy})^2]}=k\,,$$

eruimus (184 ex p. 2.^{a})

$$\left.\begin{array}{c} cos(Nx)=-\frac{1}{k}\,\frac{dz}{dx}\,,\; cos(Ny)=-\frac{1}{k}\,\frac{dz}{dy}\,, \\ cos(Nz)=\frac{1}{k}\,. \end{array}\right\}(h''')$$

110. Si aequatio ad superficiem (h') sese exhibet sub forma

$$\mu=0\ldots(h^{IV})\,,$$

cum habeamus (52. i_1)

$$\frac{dz}{dx}=-\frac{\frac{d\mu}{dx}}{\frac{d\mu}{dz}}\,,\; \frac{dz}{dy}=-\frac{\frac{d\mu}{dy}}{\frac{d\mu}{dz}}\,,$$

vertetur (h) in

$$\frac{d\mu}{dx}(v-x)+\frac{d\mu}{dy}(u-y)+\frac{d\mu}{dz}(t-z)=0\ldots(h^{V})\,;$$

et facto compendii causa

$$\sqrt{[(\frac{d\mu}{dx})^2+(\frac{d\mu}{dy})^2+(\frac{d\mu}{dz})^2]}=k'\,,$$

vertentur (h''), (h''') in

$$\left.\begin{array}{c} \frac{v-x}{\frac{d\mu}{dx}}=\frac{u-y}{\frac{d\mu}{dy}}=\frac{t-z}{\frac{d\mu}{dz}}\,, \\ cos(Nx)=\frac{1}{k'}\,\frac{d\mu}{dx}\,,\; cos(Ny)=\frac{1}{k'}\,\frac{d\mu}{dy}\,, \\ cos(Nz)=\frac{1}{k'}\,\frac{d\mu}{dz} \end{array}\right\}(h^{VI})$$

111. Ex puncto (x_0, y_0, z_0) sumpto in plano tangente duc rectam ad contactus punctum (x, y, z): aequationes ad rectam istam erunt (182. II.° ex p. 2.^{a})

$$\left\{ v - x_0 = \frac{x_0 - x}{z_0 - z}(t - z_0),\ u - y_0 = \frac{y_0 - y}{z_0 - z}(t - z_0) \right\} \ (h^{VII}).$$

Continet autem recta (h^{VII}) angulum $= 90°$ cum respondente normali; igitur (109. h'': item 183 ex p. 2.^{a})

$$1 - \frac{x_0 - x}{z_0 - z}\frac{dz}{dx} - \frac{y_0 - y}{z_0 - z}\frac{dz}{dy} = 0,$$

seu (110)

$$\frac{d\mu}{dz} + \frac{x_0 - x}{z_0 - z}\frac{d\mu}{dx} + \frac{y_0 - y}{z_0 - z}\frac{d\mu}{dy} = 0 \ldots (h^{VIII}).$$

Jam eliminatis x, y, z ex (h^{IV}), (h^{VII}) et (h^{VIII}), habitisque x_0, y_0, z_0 pro coordinatis cujusdam puncti fixi, inde obtinebitur aequatio inter v, u, t; quae cum pertineat ad rectas omnes transeuntes per (x_0, y_0, z_0), superficiemque (h^{IV}) in aliis atque aliis punctis tangentes, pertinebit etiam ad conum ipsi (h^{IV}) circumscriptum, cujus vertex in (x_0, y_0, z_0).

Ad haec: debet constanter planum tangens transire per (x_0, y_0, z_0); igitur (110. h^V)

$$\frac{d\mu}{dx}(x_0 - x) + \frac{d\mu}{dy}(y_0 - y) + \frac{d\mu}{dz}(z_0 - z) = 0 \ldots (h^{IX}).$$

Ex (h^{IV}), et (h^{IX}) eliminetur prius y v. gr., deinde x: prodibunt aequationes ad lineam illam, in qua inveniuntur omnia contactuum puncta.

112. Sint nunc

$$\left\{ v - x = a'(t - z),\ u - y = a''(t - z) \right\} \ (h^X).$$

aequationes ad rectam, quae motu sibi parallelo gi-

gnit superficiem cylindri circumscripti superficiei (h^{IV}): quoniam recta (h^{X}) continet angulum $= 90°$ cum respondente normali, iccirco (109. h'': item 183 ex p. 2.^{a})

$$\left.\begin{array}{l} 1 - a'\dfrac{dz}{dx} - a''\dfrac{dz}{dy} = 0\,, \\ \text{seu (110)} \\ \dfrac{d\mu}{dz} + a'\dfrac{d\mu}{dx} + a''\dfrac{d\mu}{dy} = 0 \end{array}\right\} (h^{\text{XI}}).$$

Jam eliminatis x, y, z ex (h^{IV}), (h^{X}) et (h^{XI}), prodibit aequatio ad circumscriptum cylindrum: aequationes (h^{IV}), (h^{XI}) praebebunt locum geometricum seu curvam, ubi inveniuntur contactuum puncta.

Exempla.

I.° Proponatur invenienda aequatio ad conum circumscriptum ellipsoidi (221 ex p. 2.^{a})

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0 \ldots (h^{\text{XII}}).$$

Ex (h^{XII}) habemus

$$\frac{d\mu}{dx} = \frac{2x}{a^2}\,,\quad \frac{d\mu}{dy} = \frac{2y}{b^2}\,,\quad \frac{d\mu}{dz} = \frac{2z}{c^2}\;;$$

proinde ex (h^{VIII})

$$\left.\begin{array}{l} \dfrac{z}{c^2}(z_0 - z) + \dfrac{x}{a^2}(x_0 - x) + \dfrac{y}{b^2}(y_0 - y) = 0\,, \\ \text{quae ob } (h^{\text{XII}}) \text{ traducitur ad} \\ \dfrac{zz_0}{c^2} + \dfrac{xx_0}{a^2} + \dfrac{yy_0}{b^2} = 1\,: \end{array}\right\} (h^{\text{XIII}})$$

in prima (h^{XIII}) substituantur valores $x_0 - x$, $y_0 - y$

ex (h^{VII}); proveniet

$$\frac{z}{c^2}(t-z_0)+\frac{x}{a^2}(v-x_0)+\frac{y}{b^2}(u-y_0)=0\,,$$

quae ob secundam (h^{XIII}) traducitur ad

$$\frac{zt}{c^2}+\frac{xv}{a^2}+\frac{yu}{b^2}=1\ \ldots\ (h^{\text{XIV}}).$$

Jamvero aequationes (h^{VII}) possunt sic scribi

$$\frac{v-x_0}{x_0-x}=\frac{u-y_0}{y_0-y}=\frac{t-z_0}{z_0-z}\,,$$

et consequenter (196 ex p. 1.^{a})

$$\frac{v-x}{x_0-x}=\frac{u-y}{y_0-y}=\frac{t-z}{z_0-z}\,;$$

igitur (192 : 197 ex p. 1.^{a})

$$\frac{(v-x)\frac{x_0}{a^2}+(u-y)\frac{y_0}{b^2}+(t-z)\frac{z_0}{c^2}}{(x_0-x)\frac{x_0}{a^2}+(y_0-y)\frac{y_0}{b^2}+(z_0-z)\frac{z_0}{c^2}}=$$

$$\frac{(v-x)\frac{v}{a^2}+(u-y)\frac{u}{b^2}+(t-z)\frac{t}{c^2}}{(x_0-x)\frac{v}{a^2}+(y_0-y)\frac{u}{b^2}+(z_0-z)\frac{t}{c^2}}\,,$$

seu ob secundam (h^{XIII}) et ob (h^{XIV})

$$\frac{\frac{vx_0}{a^2}+\frac{uy_0}{b^2}+\frac{tz_0}{c^2}-1}{\frac{x_0^2}{a^2}+\frac{y_0^2}{b^2}+\frac{z_0^2}{c^2}-1}=\frac{\frac{v^2}{a^2}+\frac{u^2}{b^2}+\frac{t^2}{c^2}-1}{\frac{vx_0}{a^2}+\frac{uy_0}{b^2}+\frac{tz_0}{c^2}-1}\,;$$

aequatio ad conum circumscriptum ellipsoidi.

II.° Invenire aequationem ad cylindrum circumscriptum ellipsoidi. Ex (h^{xi}) habemus (I.°)

$$\frac{z}{c^2}+a'\frac{x}{a^2}+a''\frac{y}{b^2}=0\ldots(h^{xv});$$

binae insuper (h^{x}) sic possunt scribi

$$t-z=\frac{v-x}{a'}=\frac{u-y}{a''}\ldots(h^{xvi}):$$

multiplicentur singuli termini (h^{xv}) per singula membra (h^{xvi}); proveniet

$$\frac{z(t-z)}{c^2}+\frac{x(v-x)}{a^2}+\frac{y(u-y)}{b^2}=0,$$

quae ob (h^{xii}) traducitur ad

$$\frac{zt}{c^2}+\frac{xv}{a^2}+\frac{yu}{b^2}=1\ldots(h^{xvii}).$$

Nunc ex (h^{xvi}) eruimus (192 : 196 : 197 ex p. 1.^{a})

$$\frac{\frac{1}{c^2}(t-z)+\frac{a'}{a^2}(v-x)+\frac{a''}{b^2}(u-y)}{\frac{1}{c^2}+\frac{a'^2}{a^2}+\frac{a''^2}{b^2}}=$$

$$\frac{\frac{t}{c^2}(t-z)+\frac{v}{a^2}(v-x)+\frac{u}{b^2}(u-y)}{\frac{t}{c^2}+\frac{a'v}{a^2}+\frac{a''u}{b^2}},$$

seu ob (h^{xv}) et (h^{xvii})

$$\frac{\frac{t}{c^2}+\frac{a'v}{a^2}+\frac{a''u}{b^2}}{\frac{1}{c^2}+\frac{a'^2}{a^2}+\frac{a''^2}{b^2}}=\frac{\frac{t^2}{c^2}+\frac{v^2}{a^2}+\frac{u^2}{b^2}-1}{\frac{t}{c^2}+\frac{a'v}{a^2}+\frac{a''u}{b^2}};$$

aequatio ad cylindrum circumscriptum ellipsoidi.

113. Quisque videt aequationes (h^{IX}), (h^{XI}) pertinere ad superficies amplectentes loca geometrica omnium contactuum; alteram in casu coni circumscripti, alteram in casu cylindri.

Fac ut (h^{IV}) recidat in (h_1. 214 ex p. 2.^{a}), spectet nimirum ad superficies secundi ordinis; vertetur (h^{IX}) in

$$(Ax+Ez+Fy+G)(x_0-x)+(By+Dz+Fx+$$
$$H)(y_0-y)+(Cz+Dy+Ex+K)(z_0-z)=0,$$

seu

$$(Ax+Ez+Fy+G)x_0+(By+Dz+Fx+H)y_0+$$
$$(Cz+Dy+Ex+K)z_0+Gx+Hy+Kz-Q=0.$$

Inferimus (180. I.° ex p. 2.^{a}), si e puncto (x_0, y_0, z_0) ducuntur plana tangentia superficiem secundi ordinis, omnia contactuum puncta fore in curva plana.

De radiis osculi diversarum curvarum quae in data superficie describi possunt; ubi et aliquid annotatur circa superficies osculatrices.

114. In data superficie (h^{IV}. 110) intelligatur describi curva; et in hac sit s arcus, r radius osculi apud punctum (x, y, z), τ tangens, ν_1 normalis principalis apud idem punctum: erit (97 : 101 : 110 : item 177 ex p. 2.^{a})

$$\cos(N\nu_{,}) = \frac{r}{k'} \cdot \frac{\frac{d\mu}{dx}d^2x + \frac{d\mu}{dy}d^2y + \frac{d\mu}{dz}d^2z}{ds^2}.$$

Ad haec: semel iterumque differentiata (h^{IV}), prodibit (49)

$$\left.\begin{array}{l} \frac{d\mu}{dx}d^2x + \frac{d\mu}{dy}d^2y + \frac{d\mu}{dz}d^2z + \frac{d^2\mu}{dx^2}dx^2 + \\ \quad \frac{d^2\mu}{dy^2}dy^2 + \frac{d^2\mu}{dz^2}dz^2 + 2\frac{d^2\mu}{dxdy}dxdy + \\ \qquad 2\frac{d^2\mu}{dxdz}dxdz + 2\frac{d^2\mu}{dydz}dydz \end{array}\right\} = 0,$$

seu (94. a^{IV})

$$\left.\begin{array}{l} \frac{\frac{d\mu}{dx}d^2x + \frac{d\mu}{dy}d^2y + \frac{d\mu}{dz}d^2z}{ds^2} + \\ \frac{d^2\mu}{dx^2}\cos^2(\tau x) + \frac{d^2\mu}{dy^2}\cos^2(\tau y) + \frac{d^2\mu}{dz^2}\cos^2(\tau z) + \\ 2\frac{d^2\mu}{dxdy}\cos(\tau x)\cos(\tau y) + 2\frac{d^2\mu}{dxdz}\cos(\tau x)\cos(\tau z) + \\ \qquad 2\frac{d^2\mu}{dydz}\cos(\tau y)\cos(\tau z) \end{array}\right\} = 0.$$

Facto igitur compendii causa

$$\left.\begin{array}{l} \frac{d^2\mu}{dx^2}\cos^2(\tau x) + \frac{d^2\mu}{dy^2}\cos^2(\tau y) + \frac{d^2\mu}{dz^2}\cos^2(\tau z) + \\ 2\frac{d^2\mu}{dxdy}\cos(\tau x)\cos(\tau y) + 2\frac{d^2\mu}{dxdz}\cos(\tau x)\cos(\tau z) + \\ \qquad 2\frac{d^2\mu}{dydz}\cos(\tau y)\cos(\tau z) \end{array}\right\} = S,$$

erit

$$cos(N\nu_1) = -\frac{r}{k'}S\text{ , ac proinde } r = -\frac{k'}{S}cos(N\nu_1)..(g).$$

Itaque data positione puncti (x, y, z), tangentis τ, et normalis ν_1, poterit inde erui radius osculi r: caeterum cum r et k' sint quantitates > 0, binae aliae S et $cos(N\nu_1)$ contrariis afficientur signis.

115. Si planum osculans (98) insistit ad perpendiculum plano tangenti, erit (77. 2° : 69. 3.° ex p. 2.ª) $cos(N\nu_1) = \pm 1$, ac proinde

$$r = \mp\frac{k'}{S}\ldots(g').$$

In ea qua sumus hypothesi cum centrum circuli osculatoris sit alicubi in N, coordinatae v, u, t illius centri explebunt primam (h^{VI}. 110): hinc (197 ex p. 1.ª)

$$\frac{(v-x)^2}{(\frac{d\mu}{dx})^2} = \frac{(u-y)^2}{(\frac{d\mu}{dy})^2} = \frac{(t-z)^2}{(\frac{d\mu}{dz})^2} =$$

$$\frac{(v-x)^2+(u-y)^2+(t-z)^2}{(\frac{d\mu}{dx})^2+(\frac{d\mu}{dy})^2+(\frac{d\mu}{dz})^2};$$

et consequenter (176 ex p. 2.ª)

$$\frac{x-v}{\frac{d\mu}{dx}} = \frac{y-u}{\frac{d\mu}{dy}} = \frac{z-t}{\frac{d\mu}{dz}} = \mp\frac{r}{k'} = \frac{1}{S}\ldots(g'').$$

116. Si aequatio ad superficiem datam ponitur resoluta quoad z, ut traducatur ad formam (h'. 108), facta $\mu = f(x, y) - z$, erunt

$$\frac{d\mu}{dx}=\frac{df}{dx}\,,\ \frac{d\mu}{dy}=\frac{df}{dy}\,,\ \frac{d\mu}{dz}=1\,,$$

$$\frac{d^2\mu}{dx^2}=\frac{d^2f}{dx^2}\,,\ \frac{d^2\mu}{dy^2}=\frac{d^2f}{dy^2}\,,\ \frac{d^2\mu}{dz^2}=0\,,$$

$$\frac{d^2\mu}{dxdy}=\frac{d^2f}{dxdy}\,,\ \frac{d^2\mu}{dxdz}=0\,,\ \frac{d^2\mu}{dydz}=0:$$

propterea

$$\left.\begin{aligned}
&k'=\sqrt{[1+(\frac{df}{dx})^2+(\frac{df}{dy})^2]},\\
&S=\frac{d^2f}{dx^2}\cos^2(\tau x)+\frac{d^2f}{dy^2}\cos^2(\tau y)+\\
&\qquad 2\frac{d^2f}{dxdy}\cos(\tau x)\cos(\tau y);\\
&\text{et } (g') \text{ vertetur in}\\
&r=\mp\frac{\sqrt{[1+(\frac{df}{dx})^2+(\frac{df}{dy})^2]}}{\frac{d^2f}{dx^2}\cos^2(\tau x)+\frac{d^2f}{dy^2}\cos^2(\tau y)+2\frac{d^2f}{dxdy}\cos(\tau x)\cos(\tau y)}
\end{aligned}\right\}(g''')$$

Jamvero (94. a^{IV}. a''')

$$\cos^2(\tau x)=\frac{dx^2}{ds^2}=\frac{dx^2}{dx^2+dy^2+df^2}=$$

$$\frac{dx^2}{dx^2+dy^2+(\frac{df}{dx}dx+\frac{df}{dy}dy)^2}=$$

$$\frac{1}{1+\frac{dy^2}{dx^2}+(\frac{df}{dx}+\frac{df}{dy}\,\frac{dy}{dx})^2},$$

$$\cos^2(\tau y)=\frac{\frac{dy^2}{dx^2}}{1+\frac{dy^2}{dx^2}+(\frac{df}{dx}+\frac{df}{dy}\frac{dy}{dx})^2},$$

$$\cos(\tau x)\cos(\tau y)=\frac{\frac{dy}{dx}}{1+\frac{dy^2}{dx^2}+(\frac{df}{dx}+\frac{df}{dy}\frac{dy}{dx})^2};$$

igitur, denotante $y=\varphi(x)$ projectionem curvae in plano XAY erunt quoque

$$\left.\begin{array}{l} S=\dfrac{\dfrac{d^2f}{dx^2}+\dfrac{d^2f}{dy^2}\varphi'^2+2\dfrac{d^2f}{dxdy}\varphi'}{1+\varphi'^2+(\dfrac{df}{dx}+\dfrac{df}{dy}\varphi')^2}, \\[2ex] r=\mp\dfrac{[1+\varphi'^2+(\dfrac{df}{dx}+\dfrac{df}{dy}\varphi')^2]\sqrt{[1+(\dfrac{df}{dx})^2+(\dfrac{df}{dy})^2]}}{\dfrac{d^2f}{dx^2}+\dfrac{d^2f}{dy^2}\varphi'^2+2\dfrac{d^2f}{dxdy}\varphi'}; \\[2ex] \text{formula } (g'') \text{ mutatur in} \\[2ex] \dfrac{x-v}{\dfrac{df}{dx}}=\dfrac{y-u}{\dfrac{df}{dy}}=z-t=\dfrac{1+\varphi'^2+(\dfrac{df}{dx}+\dfrac{df}{dy}\varphi')^2}{\dfrac{d^2f}{dx^2}+\dfrac{d^2f}{dy^2}\varphi'^2+2\dfrac{d^2f}{dxdy}\varphi'}. \end{array}\right\}(g^{\mathrm{IV}})$$

117. Proponatur nunc invenienda ejusmodi φ', cui respondeat maximus minimusve r.

Factis compendii causa

$$\left.\begin{array}{l}\frac{df}{dx}=a_1,\ \frac{df}{dy}=b_1,\ \frac{d^2f}{dx^2}=a_2,\ \frac{d^2f}{dy^2}=b_2,\ \frac{d^2f}{dxdy}=c_2,\\ \qquad =\frac{1+\varphi'^2+(a_1+b_1\varphi')^2}{a_2+b_2\varphi'^2+2c_2\varphi'}=\chi,\\ \text{exsistet}\\ \qquad r=\chi\sqrt{[1+a_1^2+b_1^2]}:\end{array}\right\}(g^{\text{v}})$$

ex penultima (g^{v}) habemus

$$\chi(a_2+b_2\varphi'^2+2c_2\varphi')\pm[1+\varphi'^2+(a_1+b_1\varphi')^2]=0\ldots(g^{\text{vi}});$$

sumptisque differentialibus quoad φ', proveniet

$$\frac{d\chi}{d\varphi'}(a_2+b_2\varphi'^2+2c_2\varphi')+2\chi(b_2\varphi'+c_2)\pm2(\varphi'+a_1b_1+b_1^2\varphi')=0.$$

Sed quoad maximos minimosve valores, quos, variata φ', recipit r, ultima (g^{v}) praebet (56)

$$\frac{dr}{d\varphi'}=\frac{d\chi}{d\varphi'}\sqrt{[1+a_1^2+b_1^2]}=0,\ \text{ideoque}\ \frac{d\chi}{d\varphi'}=0:$$

ergo

$$\left.\begin{array}{l}\chi(b_2\varphi'+c_2)\pm(\varphi'+a_1b_1+b_1^2\varphi')=0;\\ \text{et consequenter}\\ \qquad\varphi'=-\frac{c_2\chi\pm a_1b_1}{b_2\chi\pm b_1^2\pm1}.\end{array}\right\}(g^{\text{vii}})$$

Adhibita substitutione in (g^{vi}), emerget

$$(a_2\chi\pm a_1^2\pm1)(b_2\chi\pm b_1^2\pm1)-(c_2\chi\pm a_1b_1)^2=0;$$

quae, factis

$$\left.\begin{array}{l}1+a_1^2+b_1^2=\text{A},\ 2a_1b_1c_2\mp(1+a_1^2)b_2\mp\\ \qquad(1+b_1^2)a_2=\text{B},\ a_2b_2-c_2=\text{C},\end{array}\right\}(g^{\text{viii}})$$

evadet $\text{C}\chi^2-\text{B}\chi+\text{A}=0$. Hinc

$$\chi = \frac{B \pm \sqrt{[B^2-4AC]}}{2C}, \text{ et } r = \frac{B \pm \sqrt{[B^2-4AC]}}{2C} \sqrt{A} \ldots (g^{IX});$$

ex quibus valoribus alter suppeditat maximum r, alter minimum.

118. Pronum est inde concludere, si data superficies secatur planis ductis per normalem N, ex omnibus intersectionibus binas fore, alteram maxima in puncto (x, y, z) praeditam curvedine, alteram minima: istiusmodi intersectiones, nec non respondentes radii r vocantur *principales*.

Eliminata χ ex (g^{VII}) et (g^{VI}), factisque

$$\left.\begin{array}{l} \pm c_2(1+a_1^2) \mp a_1 a_2 b_1 = A_1, \; \pm a_2(1+b_1^2) \mp \\ b_2(1+a_1^2) = B_1, \; \mp c_2(1+b_1^2) \pm a_1 b_1 b_2 = C_1, \end{array}\right\} (g^X)$$

prodibit

$$C_1\varphi'^2 - B_1\varphi' + A_1 = 0, \text{ unde } \varphi' = \frac{B_1 \pm \sqrt{[B_1^2-4A_1C_1]}}{2C_1} \ldots (g^{XI}).$$

Isti duo valores φ' exprimunt (73) tangentes angulorum, quos axis AX efficit cum rectis lineis tangentibus in puncto (x, y) projectiones intersectionum principalium in plano XAY: exhibeantur ii valores per φ'_1 et φ'_2; erit

$$\varphi'_1\varphi'_2 = \frac{4A_1C_1}{4C_1^2} = \frac{A_1}{C_1}.$$

Et quoniam positio plani XAY est arbitraria, constituatur ita ut congruat cum plano tangente: rectae lineae, quibus in puncto (x, y, z) tanguntur intersectiones principales, congruent cum rectis tangentibus in puncto (x, y) projectiones ipsarum principalium intersectionum in plano XAY; eruntque apud (x, y, z)

$$z = f(x, y) = 0, \frac{df}{dx}dx + \frac{df}{dy}dy = 0,$$

et consequenter

$$\frac{df}{dx} = a_1 = 0, \frac{df}{dy} = b_1 = 0, A_1 = \pm c_2, C_1 = \mp c_2.$$

Hinc

$$\varphi'_1 \varphi'_2 + 1 = 0:$$

intersectionum videlicet principalium altera (172. I.° 4.° ex p. 2.a) exsistet alteri perpendicularis. Porro duarum curvarum altera ibi dicitur alteri perpendicularis ubi respectivae tangentes efficiunt angulum $= 90°$.

119. Superficies curvae (h'. 108) et $z = \mathrm{f}(x, y)$ ibi dicuntur sese osculari ubi et planum tangens habent commune, et sectiones principales sitas in iisdem planis normalibus praeditasque iisdem principalibus osculi radiis: itaque permanebunt φ' (g^{XI}) et respondens r (g^{IX}) quum ab una superficie transitur ad alteram. Permanet autem (108)

$$\sqrt{[1 + a_1^2 + b_1^2]};$$

idipsum ergo dicendum de χ (g^{V}), et consequenter (108) de

$$a_2 + b_2\varphi'^2 + 2c_2\varphi';$$

erit nimirum

$$\frac{d^2f}{dx^2} + \frac{d^2f}{dy^2}\varphi'^2 + 2\frac{d^2f}{dxdy}\varphi' = \frac{d^2\mathrm{f}}{dx^2} + \frac{d^2\mathrm{f}}{dy^2}\varphi'^2 + 2\frac{d^2\mathrm{f}}{dxdy}\varphi'.$$

Sume $\varphi' = 0$; habebis $\frac{d^2f}{dx^2} = \frac{d^2\mathrm{f}}{dx^2}$: proinde

$$\frac{d^2f}{dy^2}\varphi'^2 + 2\frac{d^2f}{dxdy}\varphi' = \frac{d^2\mathrm{f}}{dy^2}\varphi'^2 + 2\frac{d^2\mathrm{f}}{dxdy}\varphi',$$

$$\frac{d^2f}{dy^2}\varphi' + 2\frac{d^2f}{dxdy} = \frac{d^2\mathrm{f}}{dy^2}\varphi' + 2\frac{d^2\mathrm{f}}{dxdy}\ .$$

Sume iterum $\varphi' = 0$; habebis $\dfrac{d^2f}{dxdy} = \dfrac{d^2\mathrm{f}}{dxdy}$: sequitur quoad osculationis punctum praeter (108)

$$\left.\begin{array}{l} f = \mathrm{f},\ \dfrac{df}{dx} = \dfrac{d\mathrm{f}}{dx},\ \dfrac{df}{dy} = \dfrac{d\mathrm{f}}{dy} \\ \text{fore etiam} \\ \dfrac{d^2f}{dx^2} = \dfrac{d^2\mathrm{f}}{dx^2},\ \dfrac{d^2f}{dy^2} = \dfrac{d^2\mathrm{f}}{dy^2},\ \dfrac{d^2f}{dxdy} = \dfrac{d^2\mathrm{f}}{dxdy}\ . \end{array}\right\} (g^{\mathrm{XII}})$$

120. Mutua superficierum (h') et $z = \mathrm{f}(x, y)$ appropinquatio in viciniis puncti (x, y, z), ubi eae sese osculantur, desumi poterit ex valore infinitesimae quantitatis

$$f(x + \Delta x, y + \Delta y) - \mathrm{f}(x + \Delta x, y + \Delta y) \ldots (g^{\mathrm{XIII}}):$$

certe si ordinem infinitesimae (g^{XIII}) exprimit numerus c, superficierum (h') et $z = \mathrm{f}(x, y)$ altera ad alteram in viciniis puncti (x, y, z) accedet maxime (121 ex p. 1.[a]) aliarum omnium quibus respondet ordo ipsius (g^{XIII}) expressus per numerum $< c$.

Fac ut infinitesimae Δx, Δy sint ambae ordinis primi, et pone (39)

$$\Delta x = \beta dx,\ \Delta y = \beta dy,$$

ut (g^{XIII}) vertatur in

$$f(x + \beta dx, y + \beta dy) - \mathrm{f}(x + \beta dx, y + \beta dy);$$

hanc vero designa per $\psi(\beta)$: profecto (29), evanescente β, prima, quae inter derivatas

$$\psi'(\beta),\ \psi''(\beta),\ \psi'''(\beta), \ldots$$

non evanescit, erit ordinis vel $= c$, vel immediate

$> c$. Exsistent igitur

$$\psi(o)=0\,,\ \psi'(o)=0\,,\ \psi''(o)=0\,,\ \psi'''(o)=0\,,\ldots$$

seu (47)

$$f(x\,,\,y)-\mathrm{f}(x\,,\,y)=0\,,\ df(x\,,\,y)-d\mathrm{f}(x\,,\,y)=0\,,$$
$$d^2f(x,y)-d^2\mathrm{f}(x\,,\,y)=0\,,\ d^3f(x,y)-d^3\mathrm{f}(x,y)=0\,,\ldots$$

et consequenter

$$\left.\begin{array}{l} f(x\,,\,y)=\mathrm{f}(x\,,\,y)\,,\ df(x\,,\,y)=d\mathrm{f}(x\,,\,y)\,, \\ d^2f(x,y)=d^2\mathrm{f}(x,y)\,,\ d^3f(x,y)=d^3\mathrm{f}(x,y)\,,\ldots \end{array}\right\}(g^{\mathrm{XIV}})$$

usque ad ordinem vel $= c$, vel immediate $< c$. Prima, secunda ac tertia (g^{XIV}) praebent (46) formulas (g^{XII}) quoad osculationis punctum; quarta suppeditat (46)

$$\frac{d^3f}{dx^3}=\frac{d^3\mathrm{f}}{dx^3}\,,\ \frac{d^3f}{dy^3}=\frac{d^3\mathrm{f}}{dy^3}\,,$$
$$\frac{d^3f}{dxdy^2}=\frac{d^3\mathrm{f}}{dxdy^2}\,,\ \frac{d^3f}{dydx^2}=\frac{d^3\mathrm{f}}{dydx^2}\,;$$

et sic de caeteris.

PRINCIPIA CALCULI INTEGRALIS.

GENERALES QUAEDAM TRADUNTUR NOTIONES CIRCA INTEGRALIA INDEFINITA, NECNON CIRCA INTEGRALIA DEFINITA.

121. Quemadmodum data functione possunt quaeri ejus differentialia, ita vicissim dato differentiali quaeri potest functio unde illud promanat; et quemadmodum istarum investigationum altera spectat ad calculum differentialem, sic altera ad calculum pertinet qui dicitur *integralis*. Sint $F(x)$, $f(x)$ ejusmodi functiones variabilis x, ut existat $F'(x)=f(x)$: quantitas $F(x)+C$ vocatur *integrale indefinitum* differentialis $f(x)\,dx$, designaturque praefigendo litteram $\int$ ipsi differentiali ut scribatur

$$\int f(x)dx = F(x)+C \ldots (a);$$

exprimit C quantitatem (8) constantem atque arbitrariam.

122. Si a, b, c, ... designant quantitates constantes, et u, v, s, ... functiones variabilis x, facile (8 : 6. 2.°) intelligitur fore

$$\int audx = a\int udx, \quad \int (u+v+s+\ldots)dx = \int udx + \int vdx + \int sdx + \ldots, \quad \int (au+bv+cs+\ldots)dx = a\int udx + b\int vdx + c\int sdx + \ldots$$

123. Formula $f(x)\,dx$ ita sese aliquando exhibet, ut statim appareat eam esse differentiale cujusdam datae functionis; tunc vero in promptu est integrale: atque hoc pacto habemus (6 : 10 : 11) $\int adx = ax$,

$$\int \frac{adx}{x^2} = -\frac{a}{x}, \quad \int (a+1)x^a dx = x^{a+1}, \quad \int x^a dx =$$

$\frac{x^{a+1}}{a+1}$, $\int\frac{dx}{x^a}=-\frac{1}{(a-1)x^{a-1}}$, modo tamen in hac ultima non sit $a=1$,

$\int L(a)a^x dx = a^x$, $\int a^x dx=\frac{a^x}{L(a)}$, $\int\frac{dx}{x}=L(x)$,

$$\int cosx dx = sinx\,,\ \int sinx dx = -cosx\,,$$

$\int\frac{dx}{\sqrt{[1-x^2]}}=arc(sin=x)$, vel $=-arc(cos=x)$,

$\int\frac{dx}{1+x^2}=arc(tang=x)$, vel $=-arc(cot=x)$,

$\int\frac{dx}{x\sqrt{[x^2-1]}}=arc(sec=x)$, vel $=-arc(cosec=x)$,

$\int\frac{dx}{cos^2x}=tangx$, $\int\frac{dx}{sin^2x}=-cotx$, $\int\frac{tangx\,dx}{cosx}=$ $secx$, $\int\frac{cotx\,dx}{sinx}=-cosecx$, $\int\frac{dx}{sinx\,cosx}=L(tangx)$:

haec ultima praebet (127. 3.° ex p. 2.[a]

$\int\frac{dx}{sinx}=\int\frac{\frac{1}{2}dx}{sin\frac{1}{2}x\,cos\frac{1}{2}x}=L(tang\frac{1}{2}x)$, unde (120 : 121 ex p. 2.[a])

$$\int\frac{dx}{cosx}=\int\frac{d(\frac{\pi}{2}+x)}{sin(\frac{\pi}{2}+x)}=L(tang[\frac{\pi}{4}+\frac{x}{2}]).$$

Ad unumquodque ex his integralibus sua est addenda (121) quantitas constans et arbitraria C.

124. Interdum formula $f(x)\,dx$, de cujus integratione non constat, per quasdam substitutiones transformatur in aliam, cujus integrale illico cognoscitur. Fiat $z=\varphi(x)$, ex qua prodeat $x=\psi(z)$: erit $f(x)dx=f[\psi(z)]\psi'(z)\,dz$; id est, facta $f[\psi(z)]\psi'(z)=\chi(z)$,

$$f(x)dx=\chi(z)dz\ldots(a').$$

Jam si cognoscitur integrale formulae $\chi(z)dz$, ut sit

$$\left.\begin{array}{l}\int\chi(z)dz=\mathrm{F}(z)+\mathrm{C}, \\ \text{erit quoque} \\ \int f(x)dx=\mathrm{F}[\varphi(x)]+\mathrm{C}.\end{array}\right\}(a'')$$

Positis v. gr.

$$x^2+a^2=z,\ ax=z,\ \frac{x}{a}=z,\ x-a=z,$$

$$e^x=z,\ bx=z,\ \mathrm{L}(x)=z,\ \sin x=z,$$

$$\cos x=z,\ a+b\cos x=z,$$

emergent

$$\int\frac{xdx}{x^2+a^2}=\tfrac{1}{2}\mathrm{L}\,(x^2+a^2)+\mathrm{C},\ \int\frac{dx}{1+a^2x^2}=$$

$$\frac{1}{a}arc(tang=ax)+\mathrm{C},\ \int\frac{dx}{x^2+a^2}=\frac{1}{a}arc(tang=\frac{x}{a})+\mathrm{C},$$

$$\int\frac{dx}{x-a}=\tfrac{1}{2}\mathrm{L}(x-a)^2+\mathrm{C},\ \int\frac{dx}{(x-a)^m}=$$

$$-\frac{1}{(m-1)\,(x-a)^{m-1}}+\mathrm{C},\ \int\frac{e^xdx}{e^{2x}+1}=$$

$$arc(tang=e^x)+\mathrm{C},\ \int\cos bx\,dx=\frac{1}{b}\sin bx+\mathrm{C},$$

$$\int \frac{\mathrm{L}(x)}{x}dx = \tfrac{1}{2}\mathrm{L}^2(x) + \mathrm{C}\,,\ \int \frac{dx}{x\,\mathrm{L}(x)} = \mathrm{LL}\,(x) + \mathrm{C}\,,$$

$$\int \frac{dx}{x\mathrm{L}^m(x)} = -\frac{1}{(m-1)\mathrm{L}^{m-1}(x)} + \mathrm{C}\,,\ \int \sin x \cos x\, dx =$$

$$\frac{\sin^2 x}{2} + \mathrm{C}\,,\ \int \frac{\cos x\, dx}{\sin x} = \tfrac{1}{2}\,\mathrm{L}(\sin^2 x) + \mathrm{C}\,,$$

$$\int \frac{\sin x\, dx}{\cos x} = -\tfrac{1}{2}\,\mathrm{L}(\cos^2 x) + \mathrm{C}\,,\ \int \frac{\sin x\, dx}{a + b\cos x} =$$

$$-\frac{1}{2b}\mathrm{L}(a + b\cos x)^2 + \mathrm{C}.$$

125. In eam, quam diximus (124), transformationem nonnunquam conducit formula (9)

$$\int s dt = st - \int t ds \ldots (a''')\,;$$

sio positis 1.° $s = \dfrac{1}{\sin^2 x}$, $dt = \dfrac{dx}{\cos^2 x}$, 2.° $s = x$, $dt = \cos x\, dx$, 3.° $s = x$, $dt = \sin x\, dx$, 4.° $s = x$, $dt = a^x dx$, 5.° $s = \mathrm{L}(x)$, $dt = dx$, provenient

$$\int \frac{dx}{\sin^2 x \cos^2 x} = \frac{1}{\sin^2 x}\tan g\, x -$$

$$\int -\frac{2\tan g\, x \sin x \cos x\, dx}{\sin^4 x} = \frac{1}{\sin x \cos x} +$$

$$2\int \frac{dx}{\sin^2 x} = \frac{1}{\sin x \cos x} - 2\cot x + \mathrm{C} =$$

$$\frac{\sin^2 x + \cos^2 x}{\sin x \cos x} - 2\cot x + \mathrm{C} = \tan g\, x - \cot x + \mathrm{C}\,;$$

$$\int x \cos x\, dx = x \sin x - \int \sin x\, dx = x \sin x +$$

$\cos x + C$, $\int x \sin x\, dx = \sin x - x \cos x + C$,

$$\int x a^x dx = \frac{a^x}{L(a)}\left[x - \frac{1}{L(a)}\right] + C,\ \int L(x) dx =$$

$x(L(x) - 1) + C$.

126. Si y, z denotant binas functiones variabilis x, et ponantur

$$\int y dx = X,\ \int X \frac{dz}{dx} dx = X_1,\ \int X_1 \frac{dz}{dx} dx = X_2,$$

$$\int X_2 \frac{dz}{dx} dx = X_3, \ldots,$$

formula (a''') praebebit

$$\int y z^n dx = X z^n - n \int X z^{n-1} \frac{dz}{dx} dx,$$

$$\int X z^{n-1} \frac{dz}{dx} dx = X_1 z^{n-1} - (n-1) \int X_1 z^{n-2} \frac{dz}{dx} dx,$$

$$\int X_1 z^{n-2} \frac{dz}{dx} dx = X_2 z^{n-2} - (n-2) \int X_2 z^{n-3} \frac{dz}{dx} dx,$$

$$\int X_2 z^{n-3} \frac{dz}{dx} dx = X_3 z^{n-3} - (n-3) \int X_3 z^{n-4} \frac{dz}{dx} dx, \ldots$$

Quare

$$\left.\begin{array}{l} \int y z^n dx = X z^n - n X_1 z^{n-1} + n(n-1) X_2 z^{n-2} - \\ n(n-1)(n-2) X_3 z^{n-3} + \ldots + C. \end{array}\right\} (a^{\text{IV}})$$

127. In integrali indefinito (a) adhibeantur successive pro x peculiares valores x_0, x_n, ac dein ab $F(x_n) + C$ subtrahatur $F(x_0) + C$ ut, eliminata C, prodeat $F(x_n) - F(x_0)$: ejusmodi differentiam voco *integrale definitum* differentialis $f(x) dx$, sumptum videlicet ab $x = x_0$ ad $x = x_n$, ipsumque designo per

$\int_{x_0}^{x_n} f(x)dx$, ut scribatur

$$\int_{x_0}^{x_n} f(x)dx = \mathrm{F}(x_n) - \mathrm{F}(x^0) \ldots (a^{\mathrm{v}}).$$

Hinc v. gr.

$$\int_0^1 x^m dx = \frac{1}{m+1}, \int_0^a \frac{dx}{x^2+a^2} = \frac{\pi}{4a}.$$

128. Intervallum $x_n - x_0$ dividatur in partes, quae exhibeantur per

$$x_1 - x_0, \; x_2 - x_1, \; x_3 - x_2, \; \ldots \; x_n - x_{n-1} \; (p).$$

Quoniam (127)

$$\int_{x_0}^{x_1} f(x)dx = \mathrm{F}(x_1) - \mathrm{F}(x^0),$$

$$\int_{x_1}^{x_2} f(x)dx = \mathrm{F}(x_2) - \mathrm{F}(x_1), \ldots$$

$$\int_{x_{n-1}}^{x_n} f(x)dx = \mathrm{F}(x_n) - \mathrm{F}(x_{n-1}),$$

iccirco

$$\int_{x_0}^{x_n} f(x)dx = \int_{x_0}^{x_1} f(x)dx + \int_{x_1}^{x_2} f(x)dx +$$

$$\int_{x_2}^{x_3} f(x)dx + \ldots + \int_{x_{n-1}}^{x_n} f(x)dx.$$

Facile quoque intelligitur (122) fore

$$\int_{x_0}^{x_n}(u+v+s+\ldots)dx=\int_{x_0}^{x_n}udx+\int_{x_0}^{x_n}vdx+$$

$$\int_{x_0}^{x_n}sdx+\ldots,\ \int_{x_0}^{x_n}(au+bv+cs+\ldots)dx=$$

$$a\int_{x_0}^{x_n}udx+b\int_{x_0}^{x_n}vdx+c\int_{x_0}^{x_n}sdx+\ldots.$$

129. Numerus partium (p), quas ponimus esse ejusdem signi, indefinite augeatur, et consequenter vergant singulae ad limitem $=0$: erunt (18)

$$lim.\frac{F(x_1)-F(x_0)}{x_1-x_0}=F'(x_0)=f(x_0),$$

$$lim.\frac{F(x_2)-F(x_1)}{x_2-x_1}=f(x_1),$$

et caet. . . . ,

$$lim.\frac{F(x_n)-F(x_{n-1})}{x_n-x_{n-1}}=f(x_{n-1});$$

unde

$$\frac{F(x_1)-F(x_0)}{x_1-x_0}=f(x_0)+\sigma_1,$$

$$\frac{F(x_2)-F(x_1)}{x_2-x_1}=f(x_1)+\sigma_2,$$

et caet. . . . ,

$$\frac{F(x_n)-F(x_{n-1})}{x_n-x_{n-1}}=f(x_{n-1})+\sigma_n;$$ exprimunt

σ_1, σ_2, ... σ_n quantitates vergentes ad $lim.=0$ una cum singulis (p): hinc

$$F(x_1) - F(x_0) = (x_1 - x_0) f(x_0) + (x_1 - x_0)\sigma_1,$$

$$F(x_2) - F(x_1) = (x_2 - x_1) f(x_1) + (x_2 - x_1)\sigma_2,$$

et caet. . . . ,

$$F(x_n) - F(x_{n-1}) = (x_n - x_{n-1}) f(x_{n-1}) + (x_n - x_{n-1})\sigma_n.$$

Fiat

$$S = (x_1 - x_0) f(x_0) + (x_2 - x_1) f(x_1) +$$

$$(x_3 - x_2) f(x_2) + \ldots + (x_n - x_{n-1}) f(x_{n-1}),$$

ac denotet σ_m quantitatem mediam inter σ_1, σ_2, σ_3, . . . σ_n; erit (188 ex p. 1.a)

$$F(x_n) - F(x_0) = S + (x_n - x_0)\sigma_m.$$

Sed aucto indefinite numero partium (p), vergit σ_m ad $lim. = 0$: itaque

$$\left.\begin{aligned} & F(x_n) - F(x_0) = lim.\, S, \\ \text{idest} \quad & \\ & \int_{x_0}^{x_n} f(x)dx = lim.\, S: \end{aligned}\right\} (a^{VI})$$

superfluum est admonere de continuitate functionum F, f ab $x = x_0$ ad $x = x_n$.

Fiat etiam

$$S_1 = (x_1 - x_0) f(x_1) + (x_2 - x_1) f(x_2) + \ldots + (x_n - x_{n-1}) f(x_n);$$

exsistet

$$S_1 - S = (x_1 - x_0)[f(x_1) - f(x_0)] + (x_2 - x_1)[f(x_2) -$$

$$f(x_1)] + \ldots + (x_n - x_{n-1})[f(x_n) - f(x_{n-1})],$$

seu (188 ex p. 1.a)

$$S_1 - S = (x_n - x_0)M;$$

denotat M quantitatem mediam inter

$$f(x_1) - f(x_0),\ f(x_2) - f(x_1),\ \ldots\ f(x_n) - f(x_{n-1}).$$

Atqui hae differentiae, aucto indefinite numero partium (p), vergunt ad $lim. = 0$: idipsum ergo dicendum de media M; ideoque $lim. (S_1 - S) = 0$, et

$$lim.\, S_1 = lim. S\,,\quad \int_{x_0}^{x_n} f(x)dx = lim.\, S_1 \ldots (a^{VII}).$$

Ad haec: facto $x_0 + \varepsilon(x_n - x_0) = x_m$, formula (18)

$$\frac{F(x_n) - F(x_0)}{x_n - x_0} = f(x_0 + \varepsilon(x_n - x_0))$$

praebebit

$$\int_{x_0}^{x_n} f(x)dx = (x_n - x_0) f(x_m) \ldots (a^{VIII}).$$

130. Singula intervalla (p) intelligantur dividi in partes numero semper majores, et per ξ_0, ξ_1, ξ_2, ... ξ_{n-1} designentur quantitates mediae inter x_0 et x_1, x_1 et x_2, ..., x_{n-1} et x_n. Erunt (129. a^{VIII})

$$\int_{x_0}^{x_1} f(x)dx = (x_1 - x_0) f(\xi_0)\,,$$

$$\int_{x_1}^{x_2} f(x)dx = (x_2 - x_1) f(\xi_1)\,,$$

et caet. ...,

$$\int_{x_{n-1}}^{x_n} f(x)dx = (x_n - x_{n-1}) f(\xi_{n-1})\,;$$

unde (128)

$$\left.\begin{array}{l}\int_{x_0}^{x_n} f(x)dx = (x_1 - x_0) f(\xi_0) + \\ (x_2 - x_1) f(\xi_1) + \ldots + (x_n - x_{n-1}) f(\xi_{n-1}), \\ \text{quae potest scribi in hunc modum} \\ \int_{x_0}^{x_n} f(x)dx = (x_1 - x_0) f(x_0 + \varepsilon_0(x_1 - x_0)) + \\ (x_2 - x_1) f(x_1 + \varepsilon_1(x_2 - x_1)) + \ldots \\ + (x_n - x_{n-1}) f(x_{n-1} + \varepsilon_{n-1}(x_n - x_{n-1})); \end{array}\right\} (a^{\text{IX}})$$

denotant $\varepsilon_0, \varepsilon_1, \ldots \varepsilon_{n-1}$ numeros < 1, > 0.

131. Si partes (p) ponuntur aequales, ut sit

$$x_1 - x_0 = x_2 - x_1 = \ldots = x_n - x_{n-1} = h,$$

valores S, S_1 fient

$$S = h[f(x_0) + f(x_0 + h) + f(x_0 + 2h) + \ldots + f(x_n - h)],$$

$$S_1 = h[f(x_0 + h) + f(x_0 + 2h) + \ldots + f(x_n)]:$$

et aequatio (a^{IX}) vertetur in

$$\left.\begin{array}{l}\int_{x_0}^{x_n} f(x)dx = h[f(x_0 + \varepsilon_0 h) + \\ f(x_0 + h + \varepsilon_1 h) + f(x^0 + 2h + \varepsilon_2 h) + \\ f(x_0 + 3h + \varepsilon_3 h) + \ldots + f(x_n - h + \varepsilon_{n-1} h)]. \end{array}\right\} (a^{\text{X}})$$

Ad haec: si $f(x)$ vel constanter crescit, vel constanter decrescit ab $x = x_0$ ad $x = x_n$, valor (a^{X}) interjacebit binos S et S_1. Quocirca in ea qua sumus hypothesi, differentia inter verum integralis $\int_{x_0}^{x_n} f(x)dx$ valorem, et semi-summam

$$\frac{S+S_1}{2}=h[\tfrac{1}{2}f(x_0)+f(x_0+h)+f(x_0+2h)+\ldots$$
$$+\tfrac{1}{2}f(x_n)]\ldots(a^{XI})$$

haud pertinget ad semi-differentiam

$$\pm\tfrac{1}{2}h\,[f(x_n)-f(x_0)]$$

inter S et S_1. Formula (a^{XI}) poterit in pluribus casibus adhiberi ad obtinendum valorem vero proximum ipsius $\int_{x_0}^{x_n} f(x)dx$.

Etsi $f(x)$ neque crescit, neque decrescit constanter ab x_0 ad x_n, poterit tamen $\int_{x_0}^{x_n} f(x)dx$ decomponi (128) in plura $\int_{x_0}^{x_1} f(x)dx$, $\int_{x_1}^{x_2} f(x)dx$, ... ita, ut $f(x)$ crescat vel decrescat constanter intra singulorum limites x_0 et x_1, x_1 et x_2, ... : si quidem ipsa $f(x)$ existit continua ab x_0 ad x_n.

Utcumque se habeat $f(x)$ ab x_0 ad x_n, satis erit comparare (a^{X}) cum valoribus S et S_1 ut intelligamus differentiam vel inter S et $\int_{x_0}^{x_n} f(x)dx$, vel inter S_1 et ipsum $\int_{x_0}^{x_n} f(x)dx$ haud pertingere ad factum ex $nh=x_n-x_0$ in maximum illorum valorum, quos recipit $f(x+\Delta x)-f(x)$ quum assumitur x ab x_0 ad x_n et Δx ab 0 ad h. Est autem (21)

$$f(x+\Delta x)-f(x)=\Delta x f'(x+\varepsilon\Delta x):$$

itaque si dicatur k maximus inter valores, quorum est capax $f'(x)$ ab x_0 ad x_n, profecto differentia illa haud pertinget ad

$$hk(x_n-x_0).$$

132. Variato altero e binis limitibus v. gr. x_n in (a^{v}), variabitur ipsum quoque integrale; et adhibita x pro x_n, erit (129. a^{VIII})

$$\int_{x_0}^{x} f(x)dx = \mathrm{F}(x)-\mathrm{F}(x_0) = (x-x_0)f(x_m)\ldots(a^{\text{XII}}):$$

habebimus videlicet integrale illud, quod incipit ab x_0, quodque evanescit facto $x=x_0$: et quoniam

$$d\int_{x_0}^{x} f(x)dx = d[\mathrm{F}(x)-\mathrm{F}(x_0)] = d\mathrm{F}(x) = f(x)dx;$$

icciroo (121)

$$\int f(x)dx = \int_{x_0}^{x} f(x)dx + \mathrm{C}\ldots(a^{\text{XIII}})$$

Ex (a''') facile nunc transitur ad

$$\left.\begin{array}{l} \displaystyle\int_{x_0}^{x} \varphi(x)\psi'(x)dx = \varphi(x)\psi(x)-\varphi(x_0)\psi(x_0) - \\ \displaystyle\int_{x_0}^{x} \psi(x)\varphi'(x)dx;\ \text{item ex } (a') \text{ ad} \\ \displaystyle\int_{x_0}^{x} f(x)dx = \int_{z_0}^{z} \chi(z)dz. \end{array}\right\}(a^{\text{XIV}})$$

133. Functiones z_0, z_1, z_2, z_3, ... quantitatis variabilis x ponantur constituere seriem convergentem non solum quoad $x=x_0$ et $x=x_n$, sed etiam quoad omnes valores x interceptos inter x_0 ad x_n: sit $f(x)$ summa totius seriei, et $\chi(x)$ residuum post terminum r^{simum}, ut existat

$$f(x)=z_0+z_1+z_2+z_3+\ldots+z_{r-1}+\chi(x),$$

ideoque

$$f(x)dx=z_0dx+z_1dx+z_2dx+z_3dx+\ldots\ldots+z_{r-1}dx+\chi(x)dx.$$

Erit (128).

$$\int_{x_0}^{x_n} f(x)dx=\int_{x_0}^{x_n} z_0dx+\int_{x_0}^{x_n} z_1dx+\int_{x_0}^{x_n} z_2dx+\ldots+\int_{x_0}^{x_n} z_{r-1}dx+\int_{x_0}^{x_n}\chi(x)dx,$$

et (129. a^{VIII}) $\int_{x_0}^{x_n}\chi(x)dx=(x_n-x_0)\chi(x_m)$.

Jam vero ob positam seriei convergentiam residuum $\chi(x)$, ac proinde $\chi(x_m)$, crescente r indefinite, vergit ad *lim.* $=0$: idipsum ergo dicendum de

$\int_{x_0}^{x_n}\chi(x)\,dx$; ideoque

$$\int_{x_0}^{x_n} f(x)dx=\int_{x_0}^{x_n} z_0dx+\int_{x_0}^{x_n} z_1dx+\int_{x_0}^{x_n} z_2dx+\ldots$$

Assumptis $z_0=c_0$, $z_1=c_1x$, $z_2=c_2x^2$, $z_3=c_3x^3$, ... ut sit

$$\left.\begin{array}{l} f(x)=c_0+c_1x+c_2x^2+c_3x^3+\ldots, \\ \text{proveniet} \\ \int_{x_0}^{x_n} f(x)dx=c_0(x_n-x_0)+ \\ \frac{c_1}{2}(x^2{}_n-x^2{}_0)+\frac{c_2}{3}(x^3{}_n-x^3{}_0)+\ldots; \end{array}\right\}(a^{XV})$$

et facto $x_0=0$, adhibitoque x pro x_n,

$$\int_0^x f(x)dx=c_0x+\frac{c_1x^2}{2}+\frac{c_2x^3}{3}+\frac{c_3x^4}{4}+\ldots(a^{XVI}).$$

Hinc rursus (131) patet ratio obtinendi valores integralium veris proximos quantum libuerit: hinc quoque commoda profluit methodus varias functiones in seriem evolvendi, si nimirum per integralia definita exprimantur. Sit v. gr. in seriem evolvendus $arc(sin=x)$: sume (123)

$$arc\,(sin=x)=\int_0^x \frac{dx}{\sqrt{[1-x^2]}},$$

ac pone $x>-1$ et <1; habebis (244. ex p. 1.^{a})

$$\frac{1}{\sqrt{[1-x^2]}}=(1-x^2)^{-\frac{1}{2}}=1+\frac{x^2}{2}+$$

$$\frac{3x^4}{2.4}+\frac{3.5x^6}{2.4.6}+\frac{3.5.7x^8}{2.4.6.8}+\ldots.$$

Quocirca

$$arc(sin=x)=x+\frac{x^3}{2.3}+\frac{3x^5}{2.4.5}+\frac{3.5x^7}{2.4.6.7}+\frac{3.5.7x^9}{2.4.6.8.9}+\ldots$$

134. Sint nunc x, y binae variabiles independentes: erit (37)

$$\Delta_y \int_{x_0}^{x} f(x, y)dx = \int_{x_0}^{x} f(x, y+\Delta y)dx - \int_{x_0}^{x} f(x, y)dx,$$

seu (128)

$$\Delta_y \int_{x_0}^{x} f(x, y)dx = \int_{x_0}^{x} [f(x, y+\Delta y) -$$

$$f(x, y)]dx = \int_{x_0}^{x} \Delta_y f(x, y)\, dx.$$

Adhibita divisione per β, ac dein facto ad limites gradu, assequemur (6 : 40)

$$\left.\begin{aligned} d_y \int_{x_0}^{x} f(x, y)dx &= \int_{x_0}^{x} d_y f(x, y)dx, \\ \frac{d\int_{x_0}^{x} f(x, y)dx}{dy} &= \int_{x_0}^{x} \frac{df(x, y)}{dy} dx. \end{aligned}\right\} \quad (a^{\text{XVII}})$$

E secunda (a^{XVII}) intelligimus aequationem

$$\int_{x_0}^{x} f(x, y)dx = \chi(x, y)$$

importare sequentes

$$\int_{x_0}^{x} \frac{df(x, y)}{dy} dx = \frac{d\chi(x, y)}{dy},$$

$$\int_{x_0}^{x}\frac{d^2 f(x\,,\,y)}{dy^2}dx=\frac{d^2\chi(x\,,\,y)}{dy^2}\,,$$

et caet. . . .

$$\int_{x_0}^{x}\frac{d^n f(x\,,\,y)}{dy^n}dx=\frac{d^n\chi(x\,,\,y)}{dy^n}\,;$$

unde (132. a^{XIII})

$$\int\frac{d^n f(x\,,\,y)}{dy^n}dx=\frac{d^n\chi(x\,,\,y)}{dy^n}+C.$$

Sit v. gr. $f(x\,,\,y)=\dfrac{1}{x^2+y}$, ut habeamus (124)

$$\int_0^x\frac{dx}{x^2+y}=\chi(x\,,\,y)=\frac{1}{\sqrt{y}}\,arc\,(tang=\frac{x}{\sqrt{y}})\,:$$

quoniam

$$\frac{d(\frac{1}{x^2+y})}{dy}=-\frac{1}{(x^2+y)^2}\,,\quad\frac{d^2(\frac{1}{x^2+y})}{dy^2}=\frac{2}{(x^2+y)^3}\,,$$

$$\frac{d^3(\frac{1}{x^2+y})}{dy^3}=-\frac{2.3}{(x^2+y)^4}\,,\ldots\frac{d^n(\frac{1}{x^2+y})}{dy^n}=\pm\frac{2.3.4\ldots n}{(x^2+y)^{n+1}}\,,$$

erit igitur

$$\int_0^x\frac{d^n(\frac{1}{x^2+y})}{dy^n}dx=\pm\int_0^x\frac{2.3\ldots n\,dx}{(x^2+y)^{n+1}}=$$

$$\frac{d^n(\frac{1}{\sqrt{y}}\,arc\,(tang=\frac{x}{\sqrt{y}}))}{dy^n}\,.$$

Adhibita a pro y, ex hac formula eruitur

$$\int \frac{dx}{(x^2+a)^{n+1}} =$$

$$= \pm \frac{1}{2.3\ldots n} \, \frac{d^n\left(\frac{1}{\sqrt{a}} \operatorname{arc}\left(\operatorname{tang} = \frac{x}{\sqrt{a}}\right)\right)}{da^n} + C;$$

et simili modo possunt alia plura integralia determinari.

135. Est (127)

$$d_y \int_{y_0}^{y} f(x, y)\, dx dy = f(x, y)\, dx dy;$$

insuper ob primam (a^{xvii})

$$d_y \int_{x_0}^{x} \int_{y_0}^{y} f(x, y) dx dy = \int_{x_0}^{x} d_y \int_{y_0}^{y} f(x, y) dx dy.$$

Itaque

$$d_y \int_{x_0}^{x} \int_{y_0}^{y} f(x, y)\, dx dy = \int_{x_0}^{x} f(x, y)\, dx dy;$$

et facta integratione quoad y,

$$\int_{x_0}^{x} \int_{y_0}^{y} f(x, y) dx dy = \int_{y_0}^{y} \int_{x_0}^{x} f(x, y) dx dy \,..(a^{\text{xviii}}).$$

136. Notentur haec duo: 1.° si z exprimit datam functionem variabilis x, poniturque valere aequatio

$$\int_{x_0}^{x_n} z\varphi(x) dx = 0$$

utcumque caeteroquin se habeat functio indeterminata $\varphi(x)$, exsistet necessario $z=0$. Nisi enim exsisteret $z=0$, cum liceat ab $x=x_0$ ad $x=x_n$ sic assumere $\varphi(x)$ ut z et ipsa $\varphi(x)$ vel eodem vel contrariis signis constanter afficiantur, profecto nihil obstaret quominus $z\varphi(x)$ ab $x=x_0$ ad $x=x_n$ fieret quantitas vel constanter positiva vel constanter negativa: tunc vero facta $z\varphi(x)=\chi(x)$, integrale (129. a^{VIII})

$$\int_{x_0}^{x_n}\chi(x)dx=(x_n-x_0)\chi(x_m)$$

non esset generatim $=0$, neque proinde evanesceret generatim $\int_{x_0}^{x_n}z\varphi(x)dx$; quod cum sit contra hypothesim, exsistet necessario $z=0$. Idipsum verum erit, etsi habeantur $\varphi(x_0)=0$, $\varphi(x_n)=0$; adhuc enim intra x_0 et x_n poterit sic assumi $\varphi(x)$ ut z et ipsa $\varphi(x)$ vel eodem vel contrariis signis constanter afficiantur.

2.° posita $f(x, y)=\int_{y_0}^{y}\psi(x, y)dy$, exsistet

(132) $\dfrac{df(x, y)}{dy}=\psi(x, y)$: cum igitur (70. g^{XII})

$$\mathcal{E}((f(x, y)))=\int_{y_0}^{y}\mathcal{E}((\frac{df(x, y)}{dy}))dy,$$

erit quoque

$$\mathcal{E}((\int_{y_0}^{y}\psi(x, y)dy))=\int_{y_0}^{y}\mathcal{E}((\psi(x, y)))dy.$$

DE INTEGRATIONE DIFFERENTIALIUM ALGEBRAICORUM EX UNICA COALESCENTIUM VARIABILI.

137. Differentiale $f(x)dx$ dicitur algebraicum quotiescumque functio $f(x)$ erit (3) algebraica; poterit autem $f(x)$ esse vel rationalis, vel irrationalis. Si $f(x)$ existit rationalis, decomponetur $f(x)\,dx$ in plures terminos (71), qui sese exhibebunt sub aliqua ex hisce formis

$$bx^n dx,\ \frac{bdx}{x-k},\ \frac{bdx}{(x-k)^n},\ \frac{(a\mp b\sqrt{-1})dx}{x-k\mp h\sqrt{-1}},$$

$$\frac{(a\mp b\sqrt{-1})\,dx}{(x-k\mp h\sqrt{-1})^n};$$

denotant b, a, k, h constantes realesque quantitates, et n numerum integrum. Jam si terminorum singulorum habeantur integralia, erit (122) in promptu integrale $\int f(x)\,dx$: illorum vero terminorum integralia sunt (123: 124)

$$\int bx^n dx = \frac{bx^{n+1}}{n+1} + \mathrm{C},$$

$$\int \frac{bdx}{x-k} = \tfrac{1}{2} b\, \mathrm{L}\,(x-k)^2 + \mathrm{C},$$

$$\int \frac{bdx}{(x-k)^n} = -\frac{b}{(n-1)(x-k)^{n-1}} + \mathrm{C},$$

$$\int \frac{(a\mp b\sqrt{-1})\,dx}{x-k\mp h\sqrt{-1}} =$$

$$\int \frac{(a\mp b\sqrt{-1})(x-k\pm h\sqrt{-1})\,dx}{(x-k)^2+h^2} =$$

$$(a \mp b\sqrt{-1})\int\frac{(x-k)\,dx}{(x-k)^2+h^2}+$$

$$(b \pm a\sqrt{-1})\int\frac{h\,dx}{(x-k)^2+h^2}=$$

$$\tfrac{1}{2}(a \mp b\sqrt{-1})\,\mathrm{L}\,[(x-k)^2+h^2]+$$

$$(b \pm a\sqrt{-1})\,arc(tang=\frac{x-k}{h})+\mathrm{C},$$

$$\int\frac{(a\mp b\sqrt{-1})dx}{(x-k\mp h\sqrt{-1})^n}=$$

$$-\frac{a\mp b\sqrt{-1}}{(n-1)\,(x-k\mp h\sqrt{-1})^{n-1}}+\mathrm{C}.$$

Sic v. gr.

$$\int\frac{x^4dx}{x^2-1}=\int(x^2+1+\frac{1}{x^2-1})\,dx=$$

$$\int(x^2+1+\frac{1}{2(x-1)}-\frac{1}{2(x+1)})\,dx=$$

$$\frac{x^3}{3}+x+\frac{1}{4}\mathrm{L}(\frac{1-x}{1+x})^2+\mathrm{C},$$

$$\int\frac{dx}{x^3(x^2+1)}=\int(\frac{1}{x^3}-\frac{1}{x}+\frac{1}{2(x-\sqrt{-1})}+$$

$$\frac{1}{2(x+\sqrt{-1})})\,dx=\tfrac{1}{2}\mathrm{L}\,(\frac{x^2+1}{x^2})-\frac{1}{2x^2}+\mathrm{C}.$$

138. Functio algebraica $f(x)$ sit irrationalis; poterit adhuc adhiberi tradita methodus (137), modo sese offerat ejusmodi relatio $\chi(x, z)=0$, cujus ope differentiale $f(x)\,dx$ transformari queat in aliud rationaliter expressum per solam z.

Exempla.

I.° $$f(x)=\frac{1}{(1-x^n)\sqrt[2n]{[2x^n-1]}}$$

Relatio $$\frac{\sqrt[2n]{[2x^n-1]}}{x}=z \text{ praebet}$$

$$\frac{(1-x^n)^2}{x^{2n}}=1-z^{2n}\,,\quad \frac{(1-x^n)dx}{x^{2n+1}}=z^{2n-1}dz\,;$$

unde

$$\frac{x}{\sqrt[2n]{[2x^n-1]}}\cdot\frac{x^{2n}}{(1-x^n)^2}\cdot\frac{(1-x^n)dx}{x^{2n+1}}=$$

$$\frac{1}{z}\cdot\frac{1}{1-z^{2n}}\cdot z^{2n-1}\,dz\,,$$

seu

$$\frac{dx}{(1-x^n)\sqrt[2n]{[2x^n-1]}}=-\frac{z^{2n-2}\,dz}{z^{2n}-1}\,.$$

II.° $$f(x)=\mathrm{f}[\,x^c\,,\ \left(\frac{a'x^c+b'}{ax^c+b}\right)^{\frac{1}{p}}\,,$$

$$\left(\frac{a'x^c+b'}{ax^c+b}\right)^{\frac{1}{q}}\,,\ \ldots\,]\,x^{c-1}\,;$$

denotant p, q, ... numeros integros, c numerum quemvis, f functionem rationalem.

Assumpto numero integro m ita, ut

$$\frac{m}{p} = m', \quad \frac{m}{q} = m'', \ldots$$

existant numeri pariter integri, et facto

$$\frac{a'x^c + b'}{ax^c + b} = z^m,$$

ac proinde

$$x^c = \frac{b' - bz^m}{az^m - a'}, \quad x^{c-1}dx = \frac{mz^{m-1}(a'b - ab')dz}{c(az^m - a')^2},$$

proveniet

$$f\left[x^c, \left(\frac{a'x^c + b'}{ax^c + b}\right)^{\frac{1}{p}}, \left(\frac{a'x^c + b'}{ax^c + b}\right)^{\frac{1}{q}}, \ldots\right] x^{c-1}dx =$$

$$\frac{mz^{m-1}(a'b - ab')}{c(az^m - a')^2} f\left[\frac{b' - bz^m}{az^m - a'}, z^{m'}, z^{m''}, \ldots\right] dz.$$

Sic dato

$$\frac{x^{n-1}\,dx}{(1 - x^n)\sqrt[2n]{[2x^n - 1]}},$$

erunt $c = n$, $p = 2n$, $a = 0$, $b = 1$, $a' = 2$, $b' = -1$; et assumpto $m = 2n$ ut sit $m' = 1$, exsurgent

$$z = \sqrt[2n]{[2x^n - 1]}, \quad \frac{x^{n-1}\,dx}{(1 - x^n)\sqrt[2n]{[2x^n - 1]}} = -2\frac{z^{2n-2}dz}{z^{2n} - 1}.$$

III.° $f(x) =$

$$f(x, \sqrt{[(ax + b)^2 + (a'x + b')(a''x + b'')]}).$$

Relatio

$$(a'x+b')z^2-2(ax+b)z-(a''x+b'')=0$$

suppeditat

$$z=\frac{ax+b+\sqrt{[(ax+b)^2+(a'x+b')(a''x+b'')]}}{a'x+b'},$$

$$\sqrt{[(ax+b)^2+(a'x+b')(a''x+b'')]}=$$

$$(a'x+b')z-(ax+b),\ x=\frac{2bz-b'z^2+b''}{a'z^2-2az-a''},$$

$$dx=\frac{2[(ab'-a'b)z^2+(a''b'-a'b'')z+ab''-a''b]dz}{(a'z^2-2az-a'')^2}.$$

Quibus positis liquet formulam

$$\mathrm{f}(x,\sqrt{[(ax+b)^2+(a'x+b')(a''x+b'')]})\ dx$$

posse rationaliter exprimi per solam z.

Fiat 1.° $a=0$, $b=0$; 2.° $a'=0$, $b'=1$, $b=0$; 3.° $a'=1$, $a=0$, $b'=0$. Erit in 1.° casu

$$z=\frac{\sqrt{[(a'x+b')(a''x+b'')]}}{a'x+b'},\ \text{et}$$

$$\mathrm{f}(x,\sqrt{[(a'x+b')(a''x+b'')]})dx=$$

$$\frac{2(a''b'-a'b'')}{(a'z^2-a'')^2}z\ \mathrm{f}\left(\frac{b''-b'z^2}{a'z^2-a''},\ \frac{(a'b''-a''b')z}{a'z^2-a''}\right)dz.$$

In 2.° $z=ax+\sqrt{[a^2x^2+a''x+b'']}$, et

$$\mathrm{f}(x,\sqrt{[a^2x^2+a''x+b'']})\ dx=$$

$$\frac{2(az^2+a''z+ab'')}{(2az+a'')^2}\mathrm{f}\left(\frac{z^2-b''}{2az+a''},\ \frac{az^2+a''z+ab''}{2az+a''}\right)dz.$$

In 3.°, assumpta a'' negative,

$$z=\frac{b+\sqrt{[b^2+b''x-a''x^2]}}{x},\ \text{et}$$

$$\mathrm{f}\,(x\,,\,\sqrt{[b^2+b''x-a''x^2]})\,dx=$$

$$-\frac{2(bz^2+b''z-a''b)}{(z^2+a'')^2}\,\mathrm{f}\,(\frac{2bz+b''}{z^2+a''}\,,\,\frac{bz^2+b''z-ba''}{z^2+a''})dz.$$

Hinc v. gr.

$$\frac{dx}{\sqrt{[(a'x+b')\,(a''x+b'')]}}=-\frac{2dz}{a'z^2-a''}=$$

$$\frac{1}{\sqrt{a'a''}}(\frac{dz}{z+\sqrt{\frac{a''}{a'}}}-\frac{dz}{z-\sqrt{\frac{a''}{a'}}}),$$

$$\frac{dx}{\sqrt{[a^2x^2+a''x+b'']}}=\frac{2dz}{2az+a''}=\frac{1}{a}\cdot\frac{dz}{z+\frac{a''}{2a}}\,,$$

$$\frac{dx}{\sqrt{[b^2+b''x-a''x^2]}}=-\frac{2dz}{z^2+a''}\,;$$

et consequenter

$$\int\frac{dx}{\sqrt{[(a'x+b')(a''x+b'')]}}=$$

$$\frac{1}{\sqrt{a'a''}}\,\mathrm{L}\left(\frac{\sqrt{[a''x+b'']}+\sqrt{\frac{a''}{a'}}\sqrt{[a'x+b']}}{\sqrt{[a''x+b'']}-\sqrt{\frac{a''}{a'}}\sqrt{[a'x+b']}}\right)+\mathrm{C},$$

$$\int\frac{dx}{\sqrt{[a^2x^2+a''x+b'']}}=$$

$$\frac{\mathrm{L}(a^2x+\tfrac{1}{2}a''+a\sqrt{[a^2x^2+a''x+b'']})}{a}+\mathrm{C}\,,$$

$$\int\frac{dx}{\sqrt{[b^2+b''x-a''x^2]}}=$$

$$\frac{2}{\sqrt{a''}} arc\left(cot = \frac{b + \sqrt{[b^2 + b''x - a''x^2]}}{x\sqrt{a''}}\right) + C.$$

DE INTEGRATIONE BINOMII DIFFERENTIALIS TRADUCTA AD INTEGRATIONEM ALIORUM EJUSDEM GENERIS DIFFERENTIALIUM.

139. Formula

$$x^k (ax^c + b)^h dx \ldots (g)$$

vocatur binomium differentiale : fiat $x^c = y$; vertetur (g) in

$$\frac{1}{c} y^{\frac{k+1}{c}-1} (ay + b)^h dy \text{ seu, facto } \frac{k+1}{c} - 1 = r, \text{ in}$$

$$\frac{1}{c} y^r (ay + b)^h dy \ldots (g') :$$

itaque integratio formulae (g) traducitur ad integrationem formulae (g'). Si m, n, p denotant numeros integros, sintque vel $r = \pm m$ et $h = \pm \frac{n}{p}$, vel $r = \pm \frac{n}{p}$ et $h = \pm m$, vel $r = \pm \frac{n}{p}$ et $h = \pm m \mp \frac{n}{p}$, poterit (g') rationaliter exprimi : etenim satis erit ponere in primo casu $ay + b = z^p$, in secundo $y = z^p$, in tertio $y = (ay + b) z^p$.

140. Integratio formulae (g') traduci potest ad integrationem aliarum quarumdam formularum : fiant

$$1.° \quad s = \frac{1}{a(h+r+1)} \left(\frac{y}{ay+b}\right)^r, \quad t = (ay+b)^{h+r+1};$$

$$2.° \; s = \frac{1}{h+r+1}\left(\frac{ay+b}{y}\right)^h, \; t = y^{h+r+1};$$

$$3.° \; s = \frac{1}{b(r+1)}(ay+b)^{h+r+2}, \; t = \left(\frac{y}{ay+b}\right)^{r+1};$$

$$4.° \; s = -\frac{y^{h+r+2}}{b(h+1)}, \; t = \left(\frac{ay+b}{y}\right)^{h+1};$$

$$5.° \; s = \frac{y^r}{a(h+1)}, \; t = (ay+b)^{h+1};$$

$$6.° \; s = \frac{(ay+b)^h}{r+1}, \; t = y^{r+1};$$

erit semper $stdL(t) = y^r(ay+b)^h dy$: et quia (125. a''')

$$\int stdL(t) = st - \int stdL(s) \ldots (g''),$$

iccirco ex prima ac secunda positione emergent

$$\left.\begin{aligned} \int y^r(ay+b)^h dy &= \frac{y^r(ay+b)^{h+1}}{a(h+r+1)} - \\ &\frac{br}{a(h+r+1)}\int y^{r-1}(ay+b)^h\, dy, \\ \int y^r(ay+b)^h dy &= \frac{y^{r+1}(ay+b)^h}{h+r+1} + \\ &\frac{bh}{h+r+1}\int y^r(ay+b)^{h-1} dy; \end{aligned}\right\} (g^{\text{IV}})$$

ex tertia et quarta

$$\left.\begin{aligned}
&\int y^r(ay+b)^h dy = \frac{y^{r+1}(ay+b)^{h+1}}{b(r+1)} - \\
&\quad \frac{a(h+r+2)}{b(r+1)} \int y^{r+1}(ay+b)^h\, dy\,, \\
&\int y^r(ay+b)^h dy = -\frac{y^{r+1}(ay+b)^{h+1}}{b(h+1)} + \\
&\quad \frac{h+r+2}{b(h+1)} \int y^r(ay+b)^{h+1} dy\,;
\end{aligned}\right\} (g^{\text{IV}}),$$

ex quinta et sexta

$$\left.\begin{aligned}
&\int y^r(ay+b)^h dy = \frac{y^r(ay+b)^{h+1}}{a(h+1)} - \\
&\quad \frac{r}{a(h+1)} \int y^{r-1}(ay+b)^{h+1} dy\,, \\
&\int y^r(ay+b)^h dy = \frac{y^{r+1}(ay+b)^h}{r+1} - \\
&\quad \frac{ah}{r+1} \int y^{r+1}(ay+b)^{h-1} dy.
\end{aligned}\right\} (g^{\text{V}})$$

Sive r, h sint ambo positivi, sive ambo negativi, sive alter positivus alter negativus, paululum attendenti patebit, ope formularum (g'''), (g^{IV}), (g^{V}) traduci posse $\int y^r(ay+b)^h dy$ ad alia ejusdem generis integralia in quibus exponentes quantitatum y et $ay+b$ haud praetergrediantur limites 0, -1.

Ut istarum formularum usum binis declaremus exemplis, sit

I.° $\int \frac{dx}{(1-x^2)^m}$: erunt $k=0$, $e=2$, $h=-m$, $a=-1$, $b=1$, $r=-\frac{1}{2}$; hinc

ob secundam (g^{IV})

$$\int \frac{dx}{(1-x^2)^m} = \tfrac{1}{2} \int y^{-\frac{1}{2}} (1-y)^{-m} dy =$$

$$\frac{y^{\frac{1}{2}}}{2(m-1)(1-y)^{m-1}} + \frac{2m-3}{4(m-1)} \int y^{-\frac{1}{2}} (1-y)^{1-m} dy \,,$$

seu

$$\int \frac{dx}{(1-x^2)^m} = \frac{x}{2(m-1)(1-x^2)^{m-1}} +$$

$$\frac{2m-3}{2(m-1)} \int \frac{dx}{(1-x^2)^{m-1}} \,.$$

Rursus $\int \frac{dx}{(1-x^2)^{m-1}} = \frac{x}{2(m-2)(1-x^2)^{m-2}} +$

$$\frac{2m-5}{2(m-2)} \int \frac{dx}{(1-x^2)^{m-2}} \,, \text{ et caet.}$$

Ex quibus patet, si numerus m est integer, traduci posse $\int \frac{dx}{(1-x^2)^m}$ ad $\int dx = x + C$.

II.° $\int \frac{x^m dx}{\sqrt{[1-x^2]}}$: erunt $c=2$, $a=-1$, $b=1$, $k=m$, $h=-\frac{1}{2}$, $r=\frac{m-1}{2}$; ideoque ob primam (g'')

$$\int \frac{x^m dx}{\sqrt{[1-x^2]}} = \tfrac{1}{2} \int y^{\frac{m-1}{2}} (1-y)^{-\frac{1}{2}} dy = -$$

$$\frac{y^{\frac{m-1}{2}} (1-y)^{\frac{1}{2}}}{m} + \frac{m-1}{2m} \int y^{\frac{m-3}{2}} (1-y)^{-\frac{1}{2}} dy \,,$$

seu

$$\int \frac{x^m dx}{\sqrt{[1-x^2]}} = -\frac{x^{m-1}(1-x^2)^{\frac{1}{2}}}{m} +$$

$$\frac{m-1}{m} \int x^{m-2} (1-x^2)^{-\frac{1}{2}} dx.$$

Rursus $\int \frac{x^{m-2} dx}{\sqrt{[1-x^2]}} = -\frac{x^{m-3}(1-x^2)^{\frac{1}{2}}}{m-2} +$

$$\frac{m-3}{m-2} \int \frac{x^{m-4} dx}{\sqrt{[1-x^2]}}, \text{ et caet.}$$

Itaque si m est par, traducetur $\int \frac{x^m dx}{\sqrt{[1-x^2]}}$ ad

$$\int \frac{dx}{\sqrt{[1-x^2]}} = arc(sin = x) + C;$$

si impar, ad

$$\int \frac{x dx}{\sqrt{[1-x^2]}} = -\sqrt{[1-x^2]} + C.$$

Ex modo dictis manifeste profluunt

$$\int_0^1 \frac{x^{2n} dx}{\sqrt{[1-x^2]}} = \frac{1.3.5.7 \ldots (2n-1)}{2.4.6.8 \ldots 2n} \int_0^1 \frac{dx}{\sqrt{[1-x^2]}} =$$

$$\frac{1.3.5.7 \ldots (2n-1)}{2.4.6.8 \ldots 2n} \cdot \frac{\pi}{2}, \int_0^1 \frac{x^{2n+1} dx}{\sqrt{[1-x^2]}} =$$

$$\frac{2.4.6.8 \ldots 2n}{3.5.7.9 \ldots (2n+1)} \int_0^1 \frac{x dx}{\sqrt{[1-x^2]}} = \frac{2.4.6 \ldots 2n}{3.5.7 \ldots (2n+1)};$$

et quoniam, vergente n ad $lim. = \infty$, est

$$\lim. \int_0^1 \frac{x^{2n}dx}{\sqrt{1-x^2}} = \lim. \int_0^1 \frac{x^{2n+1}dx}{\sqrt{1-x^2}}, \text{ iccirco}$$

$$\frac{\pi}{2} = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdot \frac{8}{7} \cdot \frac{8}{9} \cdot \ldots\ldots$$

DE INTEGRATIONE DIFFERENTIALIUM COMPLECTENTIUM FUNCTIONES TRIGONOMETRICAS, LOGARITHMICAS ET EXPONENTIALES UNIUS VARIABILIS.

141. Integrale

$$\int \sin^k x \cos^g x dx \ldots (q)$$

traduci potest ad alia quaedam ejusdem generis integralia : nam facto $\sin^2 x = y$, ac proinde

$$\sin x = y^{\frac{1}{2}}, \quad \cos x = (1-y)^{\frac{1}{2}}, \quad 2 \sin x \cos x \, dx = dy,$$

$$dx = \frac{dy}{2 \sin x \cos x} = \frac{dy}{2y^{\frac{1}{2}}(1-y)^{\frac{1}{2}}}, \text{ proveniet}$$

$$(q) = \tfrac{1}{2} \int y^{\frac{k-1}{2}} (1-y)^{\frac{g-1}{2}} dy;$$

ideoque (140. $g''' \cdot g^{\text{IV}} \cdot g^{\text{V}}$)

$$\left.\begin{aligned} &\int \sin^k \cos^g x \, dx = -\frac{\sin^{k-1} x \cos^{g+1} x}{g+k} + \\ &\quad \frac{k-1}{g+k} \int \sin^{k-2} x \cos^g x \, dx, \\ &\int \sin^k \cos^g x \, dx = \frac{\sin^{k+1} x \cos^{g-1} x}{g+k} + \\ &\quad \frac{g-1}{g+k} \int \sin^k x \cos^{g-2} x \, dx, \end{aligned}\right\} (q')$$

$$\left.\begin{aligned}
\int \sin^k x \cos^g x\, dx &= \frac{\sin^{k+1} x \cos^{g+1} x}{k+1} + \\
&\quad \frac{k+g+2}{k+1} \int \sin^{k+2} x \cos^g x\, dx\,, \\
\int \sin^k x \cos^g x\, dx &= -\frac{\sin^{k+1} x \cos^{g+1} x}{g+1} + \\
&\quad \frac{k+g+2}{g+1} \int \sin^k x \cos^{g+2} x\, dx\,,
\end{aligned}\right\} (q'')$$

$$\left.\begin{aligned}
\int \sin^k x \cos^g x dx &= -\frac{\sin^{k-1} x \cos^{g+1} x}{g+1} + \\
&\quad \frac{k-1}{g+1} \int \sin^{k-2} x \cos^{g+2} x\, dx\,, \\
\int \sin^k x \cos^g x\, dx &= \frac{\sin^{k+1} x \cos^{g-1} x}{k+1} + \\
&\quad \frac{g-1}{k+1} \int \sin^{k+2} x \cos^{g-2} x\, dx.
\end{aligned}\right\} (q''')$$

Integrale (q) traduci potest ope formularum (q'), (q''), (q''') ad alia ejusdem generis integralia, in quibus exponentes quantitatum $\sin x$, $\cos x$ haud praetergrediantur limites -1, $+1$. Quod si k, g valores habeant integros, quisque videt (q) traductum iri ad aliquod ex integralibus cognitis (123 : 124)

$$\int dx\,,\ \int \sin x\, dx\,,\ \int \cos x\, dx\,,\ \int \sin x \cos x\, dx\,,$$

$$\int \frac{dx}{\sin x \cos x}\,,\ \int \frac{\sin x dx}{\cos x}\,,\ \int \frac{\cos x dx}{\sin x}\,,\ \int \frac{dx}{\sin x}\,,$$

$$\int \frac{dx}{\cos x}\,.$$

142. Facto $g=0$ in prima (q') necnon in prima (q''), in hac insuper adhibito $-k$ pro k, exsurgent

$$\left.\begin{aligned}\int \sin^k x\, dx &= -\frac{\sin^{k-1}x \cos x}{k} + \\ &\quad \frac{k-1}{k}\int \sin^{k-2}x\, dx, \\ \int \operatorname{cosec}^k x\, dx &= -\frac{\operatorname{cosec}^{k-1}x \cos x}{k-1} + \\ &\quad \frac{k-2}{k-1}\int \operatorname{cosec}^{k-2} x dx.\end{aligned}\right\} (q^{IV})$$

Facto $k=0$ in secunda (q'), necnon in secunda (q''), in hac insuper adhibito $-g$ pro g, habebimus

$$\left.\begin{aligned}\int \cos^g x\, dx &= \frac{\sin x \cos^{g-1}x}{g} + \\ &\quad \frac{g-1}{g}\int \cos^{g-2}x\, dx, \\ \int \sec^g x\, dx &= \frac{\sin x \sec^{g-1}x}{g-1} + \\ &\quad \frac{g-2}{g-1}\int \sec^{g-2}x\, dx.\end{aligned}\right\} (q^{V})$$

Substituto $-k$ pro g in prima (q''') et $-g$ pro k in secunda (q'''), exsurgent

$$\left.\begin{aligned}\int \operatorname{tang}^k x\, dx &= \frac{\operatorname{tang}^{k-1}x}{k-1} - \int \operatorname{tang}^{k-2}x\, dx, \\ \int \cot^g x\, dx &= -\frac{\cot^{g-1}x}{g-1} - \int \cot^{g-2}x\, dx.\end{aligned}\right\} (q^{VI})$$

143. Denotante n numerum integrum ac positivum, haud difficulter ex (q^{IV}), (q^{V}), (q^{VI}) eruentur pro n pari

$$\int \sin^n x\, dx = -\frac{\cos x}{n}\left[\sin^{n-1}x + \frac{n-1}{n-2}\sin^{n-3}x + \right.$$

$$\left.\frac{(n-1)(n-3)}{(n-2)(n-4)}\sin^{n-5}x + \ldots + \frac{(n-1)(n-3)\ldots 5.3}{(n-2)(n-4)\ldots 4.2}\sin x\right] +$$

$$\frac{(n-1)(n-3)\ldots 5.3.1}{n(n-2)(n-4)\ldots 4.2}x + C,$$

$$\int \operatorname{cosec}^n x dx = -\frac{\cos x}{n-1}\left[\operatorname{cosec}^{n-1}x + \right.$$

$$\frac{n-2}{n-3}\operatorname{cosec}^{n-3}x + \frac{(n-2)(n-4)}{(n-3)(n-5)}\operatorname{cosec}^{n-5}x + \ldots$$

$$\left. + \frac{(n-2)(n-4)\ldots 4.2}{(n-3)(n-5)\ldots 3.1}\operatorname{cosec} x\right] + C,$$

$$\int \cos^n x dx = \frac{\sin x}{n}\left[\cos^{n-1}x + \frac{n-1}{n-2}\cos^{n-3}x + \right.$$

$$\left.\frac{(n-1)(n-3)}{(n-2)(n-4)}\cos^{n-5}x + \ldots + \frac{(n-1)(n-3)\ldots 5.3}{(n-2)(n-4)\ldots 4.2}\cos x\right] +$$

$$\frac{(n-1)(n-3)\ldots 5.3.1}{n(n-2)(n-4)\ldots 4.2}x + C,$$

$$\int \sec^n x dx = \frac{\sin x}{n-1}\left[\sec^{n-1} x + \frac{n-2}{n-3}\sec^{n-3}x + \right.$$

$$\left.\frac{(n-2)(n-4)}{(n-3)(n-5)}\sec^{n-5}x + \ldots + \frac{(n-2)(n-4)\ldots 4.2}{(n-3)(n-5)\ldots 3.1}\sec x\right] + C,$$

$$\int tang^n x\,dx = \frac{tang^{n-1}x}{n-1} - \frac{tang^{n-3}x}{n-3} + \frac{tang^{n-5}x}{n-5} - \ldots \pm tang\,x \mp x + C,$$

$$\int cot^n x\,dx = -\frac{cot^{n-1}x}{n-1} + \frac{cot^{n-3}x}{n-3} - \frac{cot^{n-5}x}{n-5} + \ldots \pm cot\,x \pm x + C;$$

et pro n impari

$$\int sin^n x\,dx = -\frac{cos\,x}{n}[sin^{n-1}x + \frac{n-1}{n}sin^{n-3}x + \frac{(n-1)(n-3)}{(n-2)(n-4)}sin^{n-5}x + \ldots + \frac{(n-1)(n-3)\ldots 4.2}{(n-2)(n-4)\ldots 3.1}] + C,$$

$$\int cosec^n x\,dx = -\frac{cos\,x}{n-1}[cosec^{n-1}x + \frac{n-2}{n-3}cosec^{n-3}x + \frac{(n-2)(n-4)}{(n-3)(n-5)}cosec^{n-5}x + \ldots + \frac{(n-2)(n-4)\ldots 5.3}{(n-3)(n-5)\ldots 4.2}cosec^2 x] + \frac{(n-2)(n-4)\ldots 3.1}{(n-1)(n-3)\ldots 4.2}\cdot\frac{1}{2}\,L\,(tang^2\frac{x}{2}) + C,$$

$$\int cos^n x\,dx = \frac{sin\,x}{n}[cos^{n-1}x + \frac{n-1}{n-2}cos^{n-3}x + \frac{(n-1)(n-3)}{(n-2)(n-4)}cos^{n-5}x + \ldots + \frac{(n-1)(n-3)\ldots 4.2}{(n-2)(n-4)\ldots 3.1}] + C,$$

$$\int sec^n x\, dx = \frac{sin\, x}{n-1}[sec^{n-1} x + \frac{n-2}{n-3} sec^{n-3} x +$$

$$\frac{(n-2)(n-4)}{(n-3)(n-5)} sec^{n-5} x + \ldots + \frac{(n-2)(n-4)\ldots 5.3}{(n-3)(n-5)\ldots 4.2} sec^2 x] +$$

$$\frac{(n-2)(n-4)\ldots 3.1}{(n-1)(n-3)\ldots 4.2} \cdot \frac{1}{2} \mathrm{L}\,(tang^2(\frac{\pi}{4} + \frac{x}{2})) + \mathrm{C}\,,$$

$$\int tang^n x\, dx = \frac{tang^{n-1} x}{n-1} - \frac{tang^{n-3} x}{n-3} +$$

$$\frac{tang^{n-5} x}{n-5} - \ldots \pm \frac{tang^2 x}{2} \pm \tfrac{1}{2} \mathrm{L}(cos^2 x) + \mathrm{C}\,,$$

$$\int cot^n x\, dx = -\frac{cot^{n-1} x}{n-1} + \frac{cot^{n-3} x}{n-3} -$$

$$\frac{cot^{n-5} x}{n-5} + \ldots \mp \frac{cot^2 x}{2} \mp \tfrac{1}{2} \mathrm{L}(sin^2 x) + \mathrm{C}.$$

144. Si denotat f functionem algebraicam, poterit

$$\mathrm{f}\,(sin\, x\,,\ cos\, x)\, dx$$

traduci ad aliam formulam, quae omnino sit algebraice expressa. Fiat enim $sin\, x = z$, ut prodeant

$$cos\, x = \sqrt{1-z^2}\,,\ dx = \frac{dz}{\sqrt{1-z^2}} : \text{ erit}$$

$$\mathrm{f}(sin\, x\,,\ cos\, x)dx = \mathrm{f}(z\,,\ \sqrt{1-z^2})\frac{dz}{\sqrt{1-z^2}}.$$

Quod si fiat $cos\, x = z$, proveniet

$$\mathrm{f}(sin\, x\,,\ cos\, x)dx = -\mathrm{f}(\sqrt{1-z^2}\,,\ z)\frac{dz}{\sqrt{1-z^2}}.$$

Sic v. gr. haec ultima praebet

$$\frac{dx}{h+k\cos x}=-\frac{dz}{(h+kz)\sqrt{1-z^2}}.$$

Ad haec: redactis sinibus et consinibus arcuum multiplorum ad varias sinuum et cosinuum arcuum simplicium potestates (157. 2.° ex p. 2.ª), poterit etiam formula

$$f(\sin x,\ \sin 2x,\ \sin 3x,\ \ldots \cos x,\ \cos 2x,\ \cos 3x,\ \ldots)\,dx$$

ad algebraicam expressionem traduci.

145. In (a^{IV}. 126) ponantur $y=1$, $z=arc(\sin=x)$; erunt

$$X=x,\ X_1=-\sqrt{1-x^2},\ X_2=-x,\ X_3=\sqrt{1-x^2},$$
$$X_4=x,\ X_5=-\sqrt{1-x^2},\ \text{et caet.}\ldots$$

ideoque

$$\int arc^n(\sin=x)dx=x.arc^n(\sin=x)+$$
$$n\sqrt{1-x^2}.arc^{n-1}(\sin=x)-n(n-1)x.arc^{n-2}(\sin=x)-$$
$$n(n-1)(n-2)\sqrt{1-x^2}.arc^{n-3}(\sin=x)+\ldots+C.$$

Simili modo, factis $y=1$, $z=arc(\cos=x)$, assequemur

$$\int arc^n(\cos=x)dx=x.arc^n(\cos=x)-$$
$$n\sqrt{1-x^2}.arc^{n-1}(\cos=x)-n(n-1)x.arc^{n-2}(\cos=x)+$$
$$n(n-1)(n-2)\sqrt{1-x^2}.arc^{n-3}(\cos=x)+n(n-1)(n-$$
$$2)(n-3)x.arc^{n-4}(\cos=x)-\ldots+C.$$

146. In eadem formula (a^{IV}. 126) fiant $y = x^k$, $z = \mathrm{L}(x)$: erunt

$$\mathrm{X} = \frac{x^{k+1}}{k+1},\ \mathrm{X}_1 = \frac{x^{k+1}}{(k+1)^2},\ \mathrm{X}_2 = \frac{x^{k+1}}{(k+1)^3}, \ldots ;$$

unde

$$\int x^k \mathrm{L}^n(x) dx = \frac{x^{k+1}}{k+1}[\mathrm{L}^n(x) - \frac{n\mathrm{L}^{n-1}(x)}{k+1} +$$

$$\frac{n(n-1)\mathrm{L}^{n-2}(x)}{(k+1)^2} - \frac{n(n-1)(n-2)\mathrm{L}^{n-3}(x)}{(k+1)^3} + \ldots$$

$$\pm \frac{n(n-1)\ldots 3.2.1}{(k+1)^n}] + \mathrm{C}.$$

147. Habemus (16)

$$da^{(k-h\sqrt{-1})x\sqrt{-1}} =$$

$$\sqrt{-1} \,.\, \mathrm{L}(a) \,.\, (k-h\sqrt{-1})a^{(k-h\sqrt{-1})x\sqrt{-1}}dx,$$

seu

$$da^{hx}a^{kx\sqrt{-1}} = \mathrm{L}(a).(h+k\sqrt{-1})a^{hx}a^{kx\sqrt{-1}}dx;$$

et consequenter

$$\int a^{hx}a^{kx\sqrt{-1}}dx = \frac{a^{hx}a^{kx\sqrt{-1}}}{(h+k\sqrt{-1})\mathrm{L}(a)} + \mathrm{C}.$$

Jam si in (a^{IV}. 126) fiant $y = a^{hx}a^{kx\sqrt{-1}}$, $z = x$, prodibunt

$$X = \frac{a^{hx} a^{kx\sqrt{-1}}}{(h+k\sqrt{-1})L(a)},\ X_1 = \frac{a^{hx} a^{kx\sqrt{-1}}}{(h+k\sqrt{-1})^2 L^2(a)},$$

$$X_2 = \frac{a^{hx} a^{kx\sqrt{-1}}}{(h+k\sqrt{-1})^3 L^3(a)} \text{ et caet. . . . ;}$$

unde

$$\int x^n a^{hx} a^{kx\sqrt{-1}} = \frac{a^{hx} a^{kx\sqrt{-1}}}{(h+k\sqrt{-1})L(a)}[x^n -$$

$$\frac{nx^{n-1}}{(h+k\sqrt{-1})L(a)} + \frac{n(n-1)x^{n-2}}{(h+k\sqrt{-1})^2 L^2(a)} - \dots$$

$$\pm \frac{n(n-1)\dots 2.1}{(h+k\sqrt{-1})^n L^n(a)}] + C.$$

DE INTEGRALIBUS VARIORUM ORDINUM SPECTANTIBUS FORMULAM $f(x)dx^m$.

148. Habemus (134. a^{XVII})

$$d_x \int_{y_0}^{y} (x-y)^n f(y)dy = \int_{y_0}^{y} d_x (x-y)^n f(y)dy =$$

$n\int_{y_0}^{y} (x-y)^{n-1} f(y)dydx$; ideoque (135. a^{XVIII})

$$\int_{x_0}^{x}\int_{y_0}^{y}(x-y)^{n-1}f(y)dydx=\frac{1}{n}\int_{y_0}^{y}(x-y)^{n}f(y)dy;$$

et facto $n=1$, $\qquad$ (i)

$$\int_{x_0}^{x}\int_{y_0}^{y}f(y)dydx=\int_{y_0}^{y}(x-y)f(y)dy:$$

quae formulae (integralibus quoad y sumptis ab $y_0=x_0$ ad $y=x$) poterunt ita scribi

$$\left.\begin{aligned}\int_{x_0}^{x}\int_{x_0}^{x}(x-y)^{n-1}f(y)dydx&=\frac{1}{n}\int_{x_0}^{x}(x-y)^{n}f(y)dy,\\ \int_{x_0}^{x}\int_{x_0}^{x}f(y)dydx&=\int_{x_0}^{x}(x-y)f(y)dy.\end{aligned}\right\}(i')$$

His positis, fiat

$$d^m z=f(x)dx^m \ldots (i''),$$

ut determinetur z: potest (i'') exprimi in hunc modum

$$d\left(\frac{d^{m-1}z}{dx^{m-1}}\right)=f(x)dx;$$

hinc (132. a^{xIII})

$$\frac{d^{m-1}z}{dx^{m-1}}=\int f(x)dx=\int_{x_0}^{x}f(x)dx+\mathrm{C},$$

seu, quod eodem recidit, $\qquad$ (i''')

$$\frac{d^{m-1}z}{dx^{m-1}}=\int f(x)dx=\int_{x_0}^{x}f(y)dy+\mathrm{C};$$

facta videlicet integratione quoad y ab $y_0 = x_0$ ad $y = x$: et cum liceat secundam (i''') ita scribere

$$d\left(\frac{d^{m-2}z}{dx^{m-2}}\right) = \int f(x)dx^2 = \int_{x_0}^{x} f(y)dydx + Cdx,$$

erit insuper ob secundam (i')

$$\left.\begin{array}{l} \dfrac{d^{m-2}z}{dx^{m-2}} = \int\int f(x)dx^2 = \\ \int_{x_0}^{x} (x-y)f(y)dy + C(x-x_0) + C_1 \end{array}\right\} (i^{IV}).$$

Rursus (i^{IV}) potest sic disponi

$$d\left(\frac{d^{m-3}z}{dx^{m-3}}\right) = \int\int f(x)dx^3 =$$

$$\int_{x_0}^{x} (x-y)f(y)dydx + C(x-x_0)dx + C_1dx;$$

unde ob primam (i')

$$\left.\begin{array}{l} \dfrac{d^{m-3}z}{dx^{m-3}} = \int\int\int f(x)dx^3 = \\ \frac{1}{2}\int_{x_0}^{x} (x-y)^2 f(y)dy + \dfrac{C}{2}(x-x_0)^2 + C_1(x-x_0) + C_2. \end{array}\right\} (i^{V})$$

Simili modo

$$\frac{d^{m-4}z}{dx^{m-4}}=\int\int\int\int f(x)dx^4=\frac{1}{1.2.3}\int_{x_0}^{x}(x-y)^3f(y)dy+$$

$$\frac{C}{1.2.3}(x-x_0)^3+\frac{C_1}{1.2}(x-x_0)^2+C_2(x-x_0)+C_3,$$

et caet. . . .

$$\frac{dz}{dx}=\int\int\ldots f(x)dx^{m-1}=$$

$$\frac{1}{1.2.3\ldots(m-2)}\int_{x_0}^{x}(x-y)^{m-2}f(y)dy+$$

$$\frac{C}{1.2.3\ldots(m-2)}(x-x_0)^{m-2}+\frac{C_1}{1.2.3\ldots(m-3)}(x-x_0)^{m-3} \qquad (i^{\mathrm{VI}})$$

$$+\frac{C_2}{1.2.3\ldots(m-4)}(x-x_0)^{m-4}+\ldots+C_{m-2},$$

atque ultimo

$$z=\int\int\int\ldots f(x)dx^m=$$

$$\frac{1}{1.2.3\ldots(m-1)}\int_{x_0}^{x}(x-y)^{m-1}f(y)dy+$$

$$\frac{C}{1.2.3\ldots(m-1)}(x-x_0)^{m-1}+\frac{C_1}{1.2.3\ldots(m-2)}(x-x_0)^{m-2}$$

$$+\frac{C_2}{1.2.3\ldots(m-3)}(x-x_0)^{m-3}+\ldots+C_{m-1}.$$

149. Expressiones

$$dx^{m-1}\int f(x)dx,\ dx^{m-2}\int\int f(x)dx^2,$$

$$dx^{m-3}\int\int\int f(x)dx^3,\ldots\int\int\int\ldots f(x)dx^m$$

dicuntur integralia indefinita formulae differentialis $f(x)dx^m$, primi, secundi, tertii, ... m^{simi} ordinis: quod si sermo sit de integralibus definitis, disparebunt constantes arbitrariae C, C_1, C_2, ... C_{m-1} ex formulis (i'''), (i^{IV}), (i^{V}), (i^{VI}), eruntque

$$\left.\begin{array}{l}
\int_{x_0}^{x}\int_{x_0}^{x} f(x)dx^2 = \int_{x_0}^{x}(x-y)f(y)dy, \\
\int_{x_0}^{x}\int_{x_0}^{x}\int_{x_0}^{x} f(x)dx^3 = \frac{1}{1.2}\int_{x_0}^{x}(x-y)^2 f(y)dy, \\
\text{et caet. } \ldots \\
\int_{x_0}^{x}\int_{x_0}^{x}\int_{x_0}^{x}\ldots f(x)dx^m = \\
\frac{1}{1.2.3\ldots(m-1)}\int_{x_0}^{x}(x-y)^{m-1}f(y)dy.
\end{array}\right\} \quad (i^{VII})$$

Sumuntur autem in secundis membris (i^{VII}) integralia quoad y ab $y_0 = x_0$ ad $y = x$, ideoque

$$\int_{x^0}^{x}(x-y)^{m-1}f(y)dy = x^{m-1}\int_{x_0}^{x}f(y)dy -$$

$$\frac{m-1}{1}x^{m-2}\int_{x_0}^{x}yf(y)dy + \frac{(m-1)(m-2)}{1.2}x^{m-3}\times$$

$$\int_{x_0}^{x}y^2 f(y)dy - \ldots \pm \int_{x_0}^{x}y^{m-1}f(y)dy:$$

existet igitur

$$\int_{x_0}^{x}\int_{x_0}^{x}\int_{x_0}^{x}\ldots f(x)dx^m = \frac{1}{1.2.3\ldots(m-1)}[x^{m-1}\times$$

$$\int_{x_0}^{x} f(y)dy - \frac{m-1}{1}x^{m-2}\int_{x_0}^{x} yf(y)dy +$$

$$\frac{(m-1)(m-2)}{1.2}x^{m-3}\int_{x_0}^{x} y^2 f(y)dy - \ldots$$

$$\pm\int_{x_0}^{x} y^{m-1}f(y)dy\,]\ldots(i^{\mathrm{VIII}});$$

seu, quod eodem redit,

$$\int_{x_0}^{x}\int_{x_0}^{x}\int_{x_0}^{x}\ldots f(x)dx^m = \frac{1}{1.2.3\ldots(m-1)}\times$$

$$[x^{m-1}\int_{x_0}^{x} f(x)dx - \frac{m-1}{1}x^{m-2}\int_{x_0}^{x} xf(x)dx +$$

$$\frac{(m-1)(m-2)}{1.2}x^{m-3}\int_{x_0}^{x} x^2 f(x)dx - \ldots$$

$$\pm\int_{x_0}^{x} x^{m-1}f(x)dx\,]\ldots(i^{\mathrm{IX}});$$

acceptis in (i^{IX}) integralibus omnibus quoad x.

150. Exprimat $F(x)$ peculiarem valorem z, aptum videlicet aequationi (i'') adimplendae ut sit

$$F^{(m)}(x) = f(x)\ldots(i^{\mathrm{X}}).$$

In (i'''), (i^{IV}), (i^{V}), (i^{VI}) substitutis $F^{(m-1)}(x)$, $F^{(m-2)}(x)$, ... pro $\frac{d^{m-1}z}{dx^{m-1}}$, $\frac{d^{m-2}z}{dx^{m-2}}$, ... et facta

$x = x_0$, exsurgent (132. a^{XII})

$$C = F^{(m-1)}(x_0),\ C_1 = F^{(m-2)}(x_0), \ldots$$
$$C_{m-2} = F'(x_0),\ C_{m-1} = F(x_0).$$

Quare postrema (i^{VI}) evadet

$$\left.\begin{aligned}
&F(x) = F(x_0) + \frac{x-x_0}{1}F'(x_0) + \frac{(x-x_0)^2}{1.2}F''(x_0) + \\
&\frac{(x-x_0)^3}{1.2.3}F'''(x_0) + \ldots + \frac{(x-x_0)^{m-1}}{1.2.3\ldots(m-1)}F^{(m-1)}(x_0) \\
&\qquad + \frac{1}{1.2.3\ldots(m-1)}\int_{x_0}^{x}(x-y)^{m-1}F^{(m)}(y)dy\,; \\
&\text{et assumpta } x_0 = 0, \\
&F(x) = F(0) + \frac{x}{1}F'(0) + \frac{x^2}{1.2}F''(0) + \frac{x^3}{1.2.3}F'''(0) + \ldots \\
&\qquad + \frac{x^{m-1}}{1.2.3\ldots(m-1)}F^{(m-1)}(0) + \\
&\frac{1}{1.2.3\ldots(m-1)}\int_0^x (x-y)^{m-1}F^{(m)}(y)dy
\end{aligned}\right\}(i^{\text{XI}})$$

In secunda (i^{XI}) substituatur $F(x+\delta)$ pro $F(x)$, et in formula inde proveniente mutetur x in δ, ac δ in x: emerget

$$\left.\begin{aligned}
&F(x+\delta) = F(x) + \frac{\delta}{1}F'(x) + \frac{\delta^2}{1.2}F''(x) + \ldots \\
&\qquad + \frac{\delta^{m-1}}{1.2.3\ldots(m-1)}F^{(m-1)}(x) + \\
&\frac{1}{1.2.3\ldots(m-1)}\int_0^{\delta}(\delta-y)^{m-1}F^{(m)}(x+y)dy.
\end{aligned}\right\}(i^{\text{XII}})$$

Hinc (23. q^{VIII}. q^{IX} : 25. q^{XIII}. q^{XIV})

$$\frac{1}{1.2.3\ldots(m-1)}\int_0^\delta (\delta-y)^{m-1}F^{(m)}(x+y)dy =$$

$$\frac{\delta^m}{1.2.3\ldots m}F^{(m)}(x+\varepsilon\delta) = \frac{\delta^m(1-\varepsilon')^{m-1}}{1.2.3\ldots(m-1)}F^{(m)}(x+\varepsilon'\delta),$$

$$\frac{1}{1.2.3\ldots(m-1)}\int_0^x (x-y)^{m-1}F^{(m)}(y)dy =$$

$$\frac{x^m}{1.2.3\ldots m}F^{(m)}(\varepsilon x) = \frac{x^m(1-\varepsilon')^{m-1}}{1.2.3\ldots(m-1)}F^{(m)}(\varepsilon' x).$$

DE INTEGRATIONE DIFFERENTIALIUM PLURES COMPLECTENTIUM VARIABILES.

151. Quaeritur ejusmodi functio μ variabilium independentium x, y, z, t, ... qua satisfieri possit aequationi

$$\left.\begin{array}{l} d\mu = f(x, y, z, t, \ldots)dx + \varphi(x, y, z, t, \ldots)\,dy + \\ \psi(x, y, z, t, \ldots)dz + \chi(x, y, z, t, \ldots)dt + \ldots \end{array}\right\} (b),$$

seu, quod eodem redit, aequationibus (45)

$$\left.\begin{array}{l} \dfrac{d\mu}{dx} = f(x, y, z, t, \ldots), \dfrac{d\mu}{dy} = \\ \varphi(x, y, z, t, \ldots), \dfrac{d\mu}{dz} = \psi(x, y, z, t, \ldots), \\ \dfrac{d\mu}{dt} = \chi(x, y, z, t, \ldots), \text{ et caet.} \ldots \end{array}\right\} (b')$$

Ut determinatio functionis μ sit possibilis (quo

in casu differentiale (b) vocatur *exactum*), explendae sunt (43) conditiones sequentes

$$\left.\begin{aligned}
&\frac{df(x, y, z, t, \ldots)}{dy} = \frac{d\varphi(x, y, z, t, \ldots)}{dx}, \\
&\frac{df(x, y, z, t, \ldots)}{dz} = \frac{d\psi(x, y, z, t, \ldots)}{dx}, \\
&\frac{df(x, y, z, t, \ldots)}{dt} = \frac{d\chi(x, y, z, t, \ldots)}{dx}, \ldots, \\
&\frac{d\varphi(x, y, z, t, \ldots)}{dz} = \frac{d\psi(x, y, z, t, \ldots)}{dy}, \\
&\frac{d\varphi(x, y, z, t, \ldots)}{dt} = \frac{d\chi(x, y, z, t, \ldots)}{dy}, \ldots, \\
&\frac{d\psi(x, y, z, t, \ldots)}{dt} = \frac{d\chi(x, y, z, t, \ldots)}{dz}, \\
&\text{et caet.} \ldots;
\end{aligned}\right\} \quad (b'')$$

quibus expletis, ac denotante ν_1 functionem variabilium $y, z, t, \ldots$, quoniam satisfit primae (b') per

$$\mu = \int_{x_0}^{x} f(x, y, z, t, \ldots)dx + \nu_1 \ldots (b'''),$$

haec autem praebet (134. a^{XVII})

$$\frac{d\mu}{dy} = \int_{x_0}^{x} \frac{df(x, y, z, t, \ldots)}{dy} +$$

$$\frac{d\nu_1}{dy} = \int_{x_0}^{x} \frac{d\varphi(x, y, z, t, \ldots)}{dx} dx + \frac{d\nu_1}{dy} =$$

$$\varphi(x, y, z, t, \ldots) - \varphi(x_0, y, z, t, \ldots) + \frac{d\nu_1}{dy},$$

iccirco si ponitur

$$\frac{d\nu_2}{dy} - \varphi(x_0, y, z, t, \ldots) = 0,$$

et consequenter

$$\nu_1 = \int_{y_0}^{y} \varphi(x_0, y, z, t, \ldots) dy + \nu_2,$$

certe satisfiet primae ac secundae (b') per

$$\left.\begin{array}{r} \mu = \int_{x_0}^{x} f(x, y, z, t, \ldots) dx + \\ \int_{y_0}^{y} \varphi(x_0, y, z, t, \ldots) dy + \nu_2 \end{array}\right\} (b^{\text{IV}});$$

exprimit ν_2 functionem variabilium $z, t, \ldots$ Praeterea ex (b^{IV})

$$\frac{d\mu}{dz} = \int_{x_0}^{x} \frac{df(x, y, z, t, \ldots)}{dz} dx +$$

$$\int_{y_0}^{y} \frac{d\varphi(x_0, y, z, t, \ldots)}{dz} dy + \frac{d\nu_2}{dz} =$$

$$\int_{x_0}^{x} \frac{d\psi(x, y, z, t, \ldots)}{dx} dx +$$

$$\int_{y_0}^{y} \frac{d\psi(x_0, y, z, t, \ldots)}{dy} dy + \frac{d\nu_2}{dz} =$$

$$\psi(x, y, z, t, \ldots) - \psi(x_0, y_0, z, t, \ldots) + \frac{d\nu_2}{dz}.$$

Quare si ponitur

$$\frac{d\nu_2}{dz}-\psi(x_0\,,y_0\,,z\,,t\,,\ldots)=0\,,$$

et consequenter

$$\nu_2=\int_{z_0}^{z}\psi(x_0\,,y_0\,,z\,,t\,,\ldots)dz+\nu_3\,,$$

profecto satisfiet primae, secundae ac tertiae (b') per

$$\left.\begin{array}{l}\mu=\int_{x_0}^{x}f(x\,,y\,,z\,,t\,,\ldots)dx+\\ \int_{y_0}^{y}\varphi(x_0\,,y\,,z\,,t\,,\ldots)dy+\\ \int_{z_0}^{z}\psi(x_0\,,y_0\,,z\,,t\,,\ldots)dz+\nu_3\end{array}\right\}(b^{\mathrm{v}})\,;$$

exprimit ν_3 functionem variabilium $t\,,\ldots$ Rursus ex (b^{v})

$$\frac{d\mu}{dt}=\int_{x_0}^{x}\frac{df(x\,,y\,,z\,,t\,,\ldots)}{dt}dx+$$

$$\int_{y_0}^{y}\frac{d\varphi(x_0\,,y\,,z\,,t\,,\ldots)}{dt}dy+$$

$$\int_{z_0}^{z}\frac{d\psi(x_0\,,y_0\,,z\,,t\,,\ldots)}{dt}dz+\frac{d\nu_3}{dt}=$$

$$\int_{x_0}^{x}\frac{d\chi(x\,,y\,,z\,,t\,,\ldots)}{dx}dx+$$

$$\int_{y_0}^{y} \frac{d\chi(x_0, y, z, t, \ldots)}{dy} dy +$$

$$\int_{z_0}^{z} \frac{d\chi(x_0, y_0, z, t, \ldots)}{dz} dz + \frac{d\nu_3}{dt} =$$

$$\chi(x, y, z, t, \ldots) - \chi(x_0, y_0, z_0, t, \ldots) + \frac{d\nu_3}{dt}:$$

quapropter facto

$$\frac{d\nu_3}{dt} - \chi(x_0, y_0, z_0, t, \ldots) = 0,$$

ac proinde

$$\nu_3 = \int_{t_0}^{t} \chi(x_0, y_0, z_0, t, \ldots) dt + \nu_4,$$

haud dubie satisfiet primae, secundae, tertiae et quartae (b') per

$$\left.\begin{aligned} \mu = & \int_{x_0}^{x} f(x, y, z, t, \ldots) dx + \\ & \int_{y_0}^{y} \varphi(x_0, y, z, t, \ldots) dy + \\ & \int_{z_0}^{z} \psi(x_0, y_0, z, t, \ldots) dz + \\ & \int_{t_0}^{t} \chi(x_0, y_0, z_0, t, \ldots) dt + \nu_4, \end{aligned}\right\} (b^{\text{VI}}),$$

atque ita porro; denotat ν_4 functionem caeterarum variabilium post t.

Hinc si proponitur

$$d\mu=f(x,y,z)dx+\varphi(x,y,z)dy+\psi(x,y,z)dz\ldots(b^{\mathrm{VII}}),$$

adimpletis

$$\left.\begin{aligned}&\frac{df(x,y,z)}{dy}=\frac{d\varphi(x,y,z)}{dx},\ \frac{df(x,y,z)}{dz}=\\&\frac{d\psi(x,y,z)}{dx},\ \frac{d\varphi(x,y,z)}{dz}=\frac{d\psi(x,y,z)}{dy}\end{aligned}\right\}(b^{\mathrm{VIII}}),$$

erit

$$\left.\begin{aligned}\mu=&\int_{x_0}^{x}f(x,y,z)dx+\int_{y_0}^{y}\varphi(x_0,y,z)dy+\\&\int_{z_0}^{z}\psi(x_0,y_0,z)dz+\mathrm{C}.\end{aligned}\right\}(b^{\mathrm{IX}}).$$

Item si datur

$$d\mu=f(x,y)dx+\varphi(x,y)dy\ldots(b^{\mathrm{X}}),$$

expleta unica conditione

$$\frac{df(x,y)}{dy}=\frac{d\varphi(x,y)}{dx}\ldots(b^{\mathrm{XI}}),$$

erit

$$\mu=\int_{x_0}^{x}f(x,y)dx+\int_{y_0}^{y}\varphi(x_0,y)dy+\mathrm{C}\ldots(b^{\mathrm{XII}}).$$

Ut rem binis declaremus exemplis, sit

I.° $d\mu=\dfrac{ydx-xdy}{x^2+y^2}$: habemus $\dfrac{df(x,y)}{dy}=$ $\dfrac{x^2-y^2}{(x^2+y^2)^2}=\dfrac{d\varphi(x,y)}{dx}$; et quia (124) $\displaystyle\int_{x_0}^{x}f(x,y)dx=$

$$\int_{x_0}^{x} \frac{y\,dx}{x^2+y^2} = arc(tang = \frac{x}{y}) - arc(tang = \frac{x_0}{y}),$$

$$\int_{y_0}^{y} \varphi(x_0, y)dy = -\int_{y_0}^{y} \frac{x_0 dy}{x_0^2+y^2} = arc(cot = \frac{y}{x_0}) -$$

$$arc(cot = \frac{y_0}{x_0}) = arc(tang = \frac{x_0}{y}) - arc(tang = \frac{x_0}{y_0}):$$

erit igitur

$$\mu = arc(tang = \frac{x}{y}) + C.$$

II.° $d\mu = (2y^3 + 4az^2x^2)x\,dx + (\frac{1}{\sqrt{[y^2+z^2]}} +$

$3y + 2x^2)y\,dy + (4z^2 + 2ax^4 + \frac{1}{\sqrt{[y^2+z^2]}})z\,dz:$

habemus $\frac{df(x, y, z)}{dy} = 4xy = \frac{d\varphi(x, y, z)}{dx}$,

$\frac{df(x, y, z)}{dz} = 8ax^3z = \frac{d\psi(x, y, z)}{dx}$, $\frac{d\varphi(x, y, z)}{dz} =$

$-\frac{yz}{\sqrt{[y^2+z^2]}} = \frac{d\psi(x, y, z)}{dy}$; et quoniam

$$\int_{x_0}^{x} f(x, y, z)dx = x^2y^2 + ax^4z^2 - x_0^2y^2 - ax_0^4z^2,$$

$$\int_{y_0}^{y} \varphi(x_0, y, z)dy = y^3 + \sqrt{[y^2+z^2]} + x_0^2y^2 -$$

$\sqrt{[y_0^2+z^2]} - y_0^3 - x_0^2y_0^2$, $\int_{z_0}^{z} \psi(x_0, y_0, z)dz =$

$z^2 + ax^4z^3 + \sqrt{[y_0^2 + z^2]} - z_0^2 - ax_0^4z_0^3 - \sqrt{[y_0^2 + z_0^2]}$, ideo

$$\mu = x^2y^2 + ax^4z^3 + y^3 + z^4 + \sqrt{[y^2 + z^2]} + C.$$

152. Designetur nunc per V quantitas coalescens e functionibus v, u, s, ... variabilium independentium x, y, z, ... et e respectivis differentialibus dv, d^2v, ... d^nv, du, d^2u, ... d^nu, ds, d^2s, ... d^ns, et caet. ..., ut inquiramus in conditiones adimplendas ut V suis gaudeat integralibus primi, secundi, ... n^{simi} ordinis. Haec integralia vocentur S_1, S_2, S_3, ... S_n ut sint

$$VdS_1,\ S_1 = dS_2,\ S_2 = dS_3,\ \ldots\ S_{n-1} = dS_n\ \ldots\ (b^{XIII}):$$

certe in S_1 haud invenientur d^nv, d^nu, d^ns, ...; in S_2 minime reperientur $d^{n-1}v$, $d^{n-1}u$, $d^{n-1}s$, ...; atque ita usque ad S_n, in quo nullum continebitur differentiale. Veniant primo considerandae quantitates V, S_1: factis

$$\left.\begin{array}{l} dv = v_1,\ d^2v = v_2,\ d^3v = v_3,\ \ldots \\ du = u_1,\ d^2u = u_2,\ d^3u = u_3,\ \ldots \\ ds = s_1,\ d^2s = s_2,\ d^3s = s_3,\ \ldots \end{array}\right\}(b^{XIV})$$

et caet. ...

ut binarum V, S_1 altera habeatur pro functione variabilium

$$v,\ v_1,\ v_2,\ \ldots\ v_n,\ u,\ u_1,\ u_2,\ \ldots\ u_n,\ s,\ s_1,\ s_2,\ \ldots\ s_n,$$

et caet. ...,

altera pro functione

$$v,\ v_1,\ v_2\ \ldots\ v_{n-1},\ u,\ u_1,\ u_2,\ \ldots\ u_{n-1},\ s,\ s_1,\ s_2,\ \ldots\ s_{n-1},$$

et caet. ...,

licebit primam (b^{XIII}) sic exprimere

$$V=\left\{\begin{array}{l}\frac{dS_1}{dv}dv+\frac{dS_1}{dv_1}dv_1+\frac{dS_1}{dv_2}dv_2+\ldots+\frac{dS_1}{dv_{n-1}}dv_{n-1}\\+\frac{dS_1}{du}du+\frac{dS_1}{du_1}du_1+\frac{dS_1}{du_2}du_2+\ldots+\frac{dS_1}{du_{n-1}}du_{n-1}\\+\frac{dS_1}{ds}ds+\frac{dS_1}{ds_1}ds_1+\frac{dS_1}{ds_2}ds_2+\ldots+\frac{dS_1}{ds_{n-1}}ds_{n-1}\\+\ldots\ldots,\end{array}\right.$$

seu, ob (b^{xiv}),

$$V=\left\{\begin{array}{l}\frac{dS_1}{dv}v_1+\frac{dS_1}{dv_1}v_2+\frac{dS_1}{dv_2}v_3+\ldots+\frac{dS_1}{dv_{n-1}}v_n\\+\frac{dS_1}{du}u_1+\frac{dS_1}{du_1}u_2+\frac{dS_1}{du_2}u_3+\ldots+\frac{dS_1}{du_{n-1}}u_n\\+\frac{dS_1}{ds}s_1+\frac{dS_1}{ds_1}s_2+\frac{dS_1}{ds_2}s_3+\ldots+\frac{dS_1}{ds_{n-1}}s_n\\+\ldots\ldots;\end{array}\right.$$

et differentiata V successive quoad v, v_1, ... v_n, u, u_1, ... u_n, s, s_1, ... s_n, et caet. ..., prodibunt (43)

$$\left.\begin{array}{c}\frac{dV}{dv}=d\left(\frac{dS_1}{dv}\right),\ \frac{dV}{dv_1}=\frac{dS_1}{dv}+d\left(\frac{dS_1}{dv_1}\right),\\ \frac{dV}{dv_2}=\frac{dS_1}{dv_1}+d\left(\frac{dS_1}{dv_2}\right),\ \frac{dV}{dv_3}=\frac{dS_1}{dv_2}+d\left(\frac{dS_1}{dv_3}\right),\ \ldots\\ \frac{dV}{dv_{n-1}}=\frac{dS_1}{dv_{n-2}}+d\left(\frac{dS_1}{dv_{n-1}}\right),\ \frac{dV}{dv_n}=\frac{dS_1}{dv_{n-1}},\end{array}\right\}\ (b^{\text{xv}})$$

$$\left.\begin{array}{l}\frac{dV}{du}=d(\frac{dS_1}{du}),\ \frac{dV}{du_1}=\frac{dS_1}{du}+d(\frac{dS_1}{du_1}),\\ \frac{dV}{du_2}=\frac{dS_1}{du_1}+d(\frac{dS_1}{du_2}),\ \frac{dV}{du_3}=\frac{dS_1}{du_2}+d(\frac{dS_1}{du_3}),..\\ \frac{dV}{du_{n-1}}=\frac{dS_1}{du_{n-2}}+d(\frac{dS_1}{du_{n-1}}),\ \frac{dV}{du_n}=\frac{dS_1}{du_{n-1}},\end{array}\right\}(b^{xvi})$$

et caet. . . .

Quod ad (b^{xv}) pertinet, subtrahatur differentiale 2^{ae} ex 1.^{a}; erit

$$\frac{dV}{dv}-d(\frac{dV}{dv_1})=-d^2(\frac{dS_1}{dv_1}):$$

ad hanc addatur differentiale secundum 3^{ae}; proveniet

$$\frac{dV}{dv}-d(\frac{dV}{dv_1})+d^2(\frac{dV}{dv_2})=d^3(\frac{dS_1}{dv_2}):$$

cui si dematur differentiale tertium 4^{ae}, exsurget

$$\frac{dV}{dv}-d(\frac{dV}{dv_1})+d^2(\frac{dV}{dv_2})-d^3(\frac{dV}{dv_3})=-d^4(\frac{dS_1}{dv_3});$$

atque sic progrediendo devenietur tandem ad

$$\left.\begin{array}{l}\frac{dV}{dv}-d(\frac{dV}{dv_1})+d^2(\frac{dV}{dv_2})-d^3(\frac{dV}{dv_3})+\\ d^4(\frac{dV}{dv_4})-\ldots\pm d^n(\frac{dV}{dv_n})=0;\\ \text{simili modo ex } (b^{xvi}), \text{ et caet. . . .}\\ \frac{dV}{du}-d(\frac{dV}{du_1})+d^2(\frac{dV}{du_2})-d^3(\frac{dV}{du^3})+\ldots\pm d^n(\frac{dV}{du_n})=0,\\ \frac{dV}{ds}-d(\frac{dV}{ds_1})+d^2(\frac{dV}{ds_2})-d^3(\frac{dV}{ds_3})+\ldots\pm d^n(\frac{dV}{ds_n})=0,\end{array}\right\}(I)$$

et caet. . . .

valet signum superius si n est par, inferius si impar: facto $n=1$, et habitis $v, u, s, \ldots$ pro ipsis variabilibus independentibus $x, y, z, \ldots$, ex formulis (I) facili eruentur negotio conditiones jam assignatae (151. b'').

Quemadmodum prima (b^{XIII}. 152) importat aequationes (I), ita secunda inter easdem (b^{XIII}) importat sequentes

$$\left.\begin{aligned}
&\frac{dS_1}{dv}-d\left(\frac{dS_1}{dv_1}\right)+d^2\left(\frac{dS_1}{dv_2}\right)-d^3\left(\frac{dS_1}{dv_3}\right)+\ldots\\
&\qquad\mp d^{n-1}\left(\frac{dS_1}{dv_{n-1}}\right)=0,\\
&\frac{dS_1}{du}-d\left(\frac{dS_1}{du_1}\right)+d^2\left(\frac{dS_1}{du_2}\right)-d^3\left(\frac{dS_1}{du_3}\right)+\ldots\\
&\qquad\mp d^{n-1}\left(\frac{dS_1}{du_{n-1}}\right)=0,\\
&\frac{dS_1}{ds}-d\left(\frac{dS_1}{ds_1}\right)+d^2\left(\frac{dS_1}{ds_2}\right)-d^3\left(\frac{dS_1}{ds_3}\right)+\ldots\\
&\qquad\mp d^{n-1}\left(\frac{dS_1}{ds_{n-1}}\right)=0,
\end{aligned}\right\}(b^{\text{XVII}})$$

et caet. ....

Jam vero secunda, tertia, ... (b^{XV}) praebent

$$\left.\begin{aligned}
&\frac{dS_1}{dv}=\frac{dV}{dv_1}-d\left(\frac{dS_1}{dv_1}\right),\ d\left(\frac{dS_1}{dv_1}\right)=d\left(\frac{dV}{dv_2}\right)-d^2\left(\frac{dS_1}{dv_2}\right),\\
&\qquad d^2\left(\frac{dS_1}{dv_2}\right)=d^2\left(\frac{dV}{dv_3}\right)-d^3\left(\frac{dS_1}{dv_3}\right),\ \ldots\\
&d^{n-2}\left(\frac{dS_1}{dv_{n-2}}\right)=d^{n-2}\left(\frac{dV}{dv_{n-1}}\right)-d^{n-1}\left(\frac{dS_1}{dv_{n-1}}\right),\\
&\qquad d^{n-1}\left(\frac{dS_1}{dv_{n-1}}\right)=d^{n-1}\left(\frac{dV}{dv_n}\right);
\end{aligned}\right\}(b^{\text{XVIII}})$$

hinc

$$\frac{dS_1}{dv} = \frac{dV}{dv_1} - d\left(\frac{dV}{dv_2}\right) + d^2\left(\frac{dV}{dv_3}\right) - \ldots \mp d^{n-1}\left(\frac{dV}{dv_n}\right),$$

$$d\left(\frac{dS_1}{dv_1}\right) = d\left(\frac{dV}{dv_2}\right) - d^2\left(\frac{dV}{dv_3}\right) + d^3\left(\frac{dV}{dv_4}\right) - \ldots \mp d^{n-1}\left(\frac{dV}{dv_n}\right),$$

$$d^2\left(\frac{dS_1}{dv_2}\right) = d^2\left(\frac{dV}{dv_3}\right) - d^3\left(\frac{dV}{dv_4}\right) + \ldots \mp d^{n-1}\left(\frac{dV}{dv_n}\right),$$

et caet. . . .

Adhibitis substitutionibus in prima (b^{xvii}), exsurget

$$\left.\begin{array}{l}
\frac{dV}{dv_1} - 2d\left(\frac{dV}{dv_2}\right) + 3d^2\left(\frac{dV}{dv_3}\right) - 4d^3\left(\frac{dV}{dv_4}\right) + \ldots \\
\qquad\qquad \mp nd^{n-1}\left(\frac{dV}{dv_n}\right) = 0; \\
\text{et simili modo} \\
\frac{dV}{du_1} - 2d\left(\frac{dV}{du_2}\right) + 3d^2\left(\frac{dV}{du_3}\right) - 4d^3\left(\frac{dV}{du_4}\right) + \ldots \\
\qquad\qquad \mp nd^{n-1}\left(\frac{dV}{du_n}\right) = 0, \\
\frac{dV}{ds_1} - 2d\left(\frac{dV}{ds_2}\right) + 3d^2\left(\frac{dV}{ds_3}\right) - \ldots \mp nd^{n-1}\left(\frac{dV}{ds_n}\right) = 0, \\
\text{et caet.} \ldots
\end{array}\right\} \text{(II)}$$

Sicuti secunda et prima (b^{xiii}) important aequationes (II), sic tertia et secunda important sequentes

$$\frac{dS_1}{dv_1} - 2d\left(\frac{dS_1}{dv_2}\right) + 3d^2\left(\frac{dS_1}{dv_3}\right) - 4d^3\left(\frac{dS_1}{dv_4}\right) + \ldots$$

$$\pm(n-1)d^{n-2}\left(\frac{dS_1}{dv_{n-1}}\right)=0,$$

$$\frac{dS_1}{du_1}-2d\left(\frac{dS_1}{du_2}\right)+3d^2\left(\frac{dS_1}{du_3}\right)-\ldots\pm(n-1)d^{n-2}\left(\frac{dS_1}{du_{n-1}}\right)=0,$$

$$\frac{dS_1}{ds_1}-2d\left(\frac{dS_1}{ds_2}\right)+3d^2\left(\frac{dS_1}{ds_3}\right)-\ldots\pm(n-1)d^{n-2}\left(\frac{dS_1}{ds_{n-1}}\right)=0,$$

et caet. . . .

seu, ob (b^{xv}),

$$\left.\begin{array}{l}
\frac{dV}{dv_2}-3d\left(\frac{dV}{dv_3}\right)+6d^2\left(\frac{dV}{dv_4}\right)-10d^3\left(\frac{dV}{dv_5}\right)+\ldots \\
\qquad\qquad \pm\frac{n(n-1)}{2}d^{n-2}\left(\frac{dV}{dv_n}\right)=0; \\
\text{itemque} \\
\frac{dV}{du_2}-3d\left(\frac{dV}{du_3}\right)+6d^2\left(\frac{dV}{du_4}\right)-10d^3\left(\frac{dV}{du_5}\right)+\ldots \\
\qquad\qquad \pm\frac{n(n-1)}{2}d^{n-2}\left(\frac{dV}{du_n}\right)=0, \\
\frac{dV}{ds_2}-3d\left(\frac{dV}{ds_3}\right)+6d^2\left(\frac{dV}{ds_4}\right)-\ldots\pm\frac{n(n-1)}{2}d^{n-2}\left(\frac{dV}{ds_n}\right)=0, \\
\text{et caet.}\ldots
\end{array}\right\}\text{(III)}$$

Quarta, tertia et secunda (b^{xiii}) important sequentes

$$\frac{dS_1}{dv_2}-3d\left(\frac{dS_1}{dv_3}\right)+6d^2\left(\frac{dS_1}{dv_4}\right)-10d^3\left(\frac{dS_1}{dv_5}\right)+\ldots$$

$$\mp\frac{(n-1)(n-2)}{2}d^{n-3}\left(\frac{dS_1}{dv_{n-1}}\right)=0,$$

$$\frac{dS_1}{du_2}-3d(\frac{dS_1}{du_3})+6d^2(\frac{dS_1}{du_4})-\ldots\mp\frac{(n-1)(n-2)}{2}d^{n-3}(\frac{dS_1}{du_{n-1}})=0,$$

$$\frac{dS_1}{ds_2}-3d(\frac{dS_1}{ds_3})+6d^2(\frac{dS_1}{ds_4})-\ldots\mp\frac{(n-1)(n-2)}{2}d^{n-3}(\frac{dS_1}{ds_{n-1}})=0,$$

et caet.... ;

seu, ob (b^{xv}),

$$\left.\begin{array}{l}
\frac{dV}{dv_3}-4d(\frac{dV}{dv_4})+10d^2(\frac{dV}{dv_5})-20d^3(\frac{dV}{dv_6})+\ldots \\
\qquad\mp\frac{n(n-1)(n-2)}{2.3}d^{n-3}(\frac{dV}{dv_n})=0, \\
\frac{dV}{du_3}-4d(\frac{dV}{du_4})+10d^2(\frac{dV}{du_5})-20d^3(\frac{dV}{du_6})+\ldots \\
\qquad\mp\frac{n(n-1)(n-2)}{2.3}d^{n-3}(\frac{dV}{du_n})=0, \\
\frac{dV}{ds_3}-4d(\frac{dV}{ds_4})+10d^2(\frac{dV}{ds_5})-\ldots \\
\qquad\mp\frac{n(n-1)(n-2)}{2.3}d^{n-3}(\frac{dV}{ds_n})=0,
\end{array}\right\}\text{(IV)}$$

et caet....

Superfluum est longius progredi in hujusmodi inquisitionibus, cum lex formularum (I), (II), ... sit manifesta: interea ut unum praebeamus exemplum, detur

$$V=6udv^2+12vdvdu+3v^2d^2u+6vud^2v;$$

erit $n=2$, et V transformabitur in

$$V=6uv_1^2+12vv_1u_1+3v^2u_2+6vuv_2.$$

Aequationes (I) rediguntur ad

$$\frac{dV}{dv}-d\left(\frac{dV}{dv_1}\right)+d^2\left(\frac{dV}{dv_2}\right)=0, \quad \frac{dV}{du}-d\left(\frac{dV}{du_1}\right)+d^2\left(\frac{dV}{du_2}\right)=0;$$

aequationes (II) ad

$$\frac{dV}{dv_1}-2d\left(\frac{dV}{dv_2}\right)=0, \quad \frac{dV}{du_1}-2d\left(\frac{dV}{du_2}\right)=0;$$

quibus satisfaciunt valores

$$\frac{dV}{dv}=12v_1u_1+6vu_2+6uv_2, \quad \frac{dV}{dv_1}=12uv_1+12vu_1,$$

$$d\left(\frac{dV}{dv_1}\right)=24v_1u_1+12uv_2+12vu_2, \quad d\left(\frac{dV}{dv_2}\right)=6vu_1+6uv_1,$$

$$d^2\left(\frac{dV}{dv_2}\right)=12v_1u_1+6vu_2+6uv_2, \quad \frac{dV}{du}=6v_1^2+6vu_2,$$

$$\frac{dV}{du_1}=12vv_1, \quad d\left(\frac{dV}{du_1}\right)=12vv_2+12v_1^2,$$

$$d\left(\frac{dV}{du_2}\right)=6vv_1, \quad d^2\left(\frac{dV}{du_2}\right)=6vv_2+6v_1^2.$$

DE INTEGRATIONE AEQUATIONUM DIFFERENTIALIUM PRIMI ORDINIS, UBI ET ALIQUID ANNOTATUR CIRCA SOLUTIONES PARTICULARES.

153. Sit

$$Mdx+Ndy=0 \ldots (o)$$

aequatio differentialis primi ordinis inter binas variabiles; designant M, N functiones variabilium x, y. Si

$$\frac{dM}{dy}=\frac{dN}{dx},$$

methodus integrandi aequationem (o) erit ipsa eadem,

quam exposuimus (151. b^{XII}) : sin minus, oportebit aut separare variabiles x, y, idest aequationem (o) traducere ad formam

$$Xdx+Ydy=0 \ldots (o');$$

aliamve huic similem (denotat X functionem solius x, et Y functionem solius y) ut habeatur

$$\frac{dX}{dy}=0=\frac{dY}{dx};$$

aut ejusmodi functionem φ variabilium x, y invenire, per quam multiplicata (o), sicque habita

$$\varphi Mdx+\varphi Ndy=0 \ldots (o''),$$

inde prodeat

$$\frac{d(\varphi M)}{dy}=\frac{d(\varphi N)}{dx} \ldots (o'''):$$

etenim ex jam traditis de integratione functionum eruetur vel integrale aequationis (o'), vel integrale aequationis (o''); quae integralia manifeste pertinebunt etiam ad (o).

154. Separationem variabilium absque ulla difficultate assequimur 1.° si, denotante X_1 functionem solius x, et Y_1 functionem solius y, habeatur $\frac{M}{N}=X_1 Y_1$; nam (o) manifeste vertetur in

$$X_1dx+\frac{dy}{Y_1}=0;$$

2.° si exponentes variabilium x, y in singulis terminis functionum M, N eamdem summam efficiunt, ideoque M, N homogeneae (64): assumpta enim $y=xz$, et consequenter $dy=xdz+zdx$, cum $\frac{M}{N}$ evadat functio Z solius z, transformabitur (o) in

$$(Z+z)dx+xdz=0\text{, unde } \frac{dx}{x}+\frac{dz}{Z+z}=0.$$

155. In aliis casibus vel nullo pacto conceditur separare variabiles, vel si aliqua est via id obtinendi, ea in quibusdam substitutionibus consistit. Etsi nulla certa methodus tradi potest, per quam liceat cognoscere utrum variabiles possint separari, et quaenam substitutio apta sit obtinendae separationi; attamen non desunt artificia analytica, quibus rite adhibitis haud raro sese offert substitutio ad separandas variabiles opportuna. Primum artificium est ejusmodi: quae quantitates integrationem impediunt, eae reijciantur in communes multiplicatores ac divisores; dein caeterarum integrale aequetur novae variabili, et ope substitutionis eliminetur altera ex variabilibus ab aequatione data. Hoc pacto saepe contingit ut in nova aequatione variabiles existant separatae.

Exempla.

I.° Detur

$$\frac{xydy+y^2dx}{x^2y^2+a^4}=\frac{dY}{a^2},$$

quae potest scribi in hunc modum

$$\frac{y}{x^2y^2+a^4}d(xy)=\frac{dY}{a^2}.$$

Facto $xy=az$, ideoque $d(xy)=adz$, $x=\frac{az}{y}$, exsurget

$$\frac{aydz}{a^2z^2+a^4}=\frac{dY}{a^2};\quad\text{et}\quad\frac{dY}{y}-\frac{adz}{z^2+a^2}=0,$$

aequatio cum variabilibus separatis.

II.° Detur

$$bydx - \frac{a^3dx}{x} = aydy ,$$

quae disponatur ita

$$y(bdx - ady) = \frac{a^3dx}{x}, \text{ seu } yd(bx - ay) = \frac{a^3dx}{x} .$$

Facto $bx - ay = az$, ut sint

$$y = \frac{bx - az}{a},\ bdx - ady = adz ,$$

proveniet

$$bxdz - azdz = \frac{a^3dx}{x} .$$

Sume $zdz = \frac{a^2ds}{s}$; habebis

$$bxdz = a^3(\frac{ds}{s} + \frac{dx}{x}) = a^3\frac{d(xs)}{xs} .$$

Fiat $xs = v$, et consequenter $x = \frac{v}{s}$; erit

$$\frac{bvdz}{s} = a^3\frac{dv}{v}, \text{ seu } \frac{bdz}{s} - a^3\frac{dv}{v^2} = 0 ;$$

in qua cum s data sit per z, variabiles exsistent separatae.

III.° Sit

$$\frac{2ydy + xdy + ydx}{a + x + y} = dY ,$$

quae potest sic exprimi

$$\frac{d(y^2 + xy)}{a + x + y} = dY.$$

Posito $y^2+xy=az$, ac proinde

$$2ydy+xdy+ydx=adz\,,\; x=\frac{az-y^2}{y}\,,$$

exsurget

$$\frac{adz}{a+\frac{az-y^2}{y}+y}=dY\,,\ \text{seu}\ \frac{ydz}{y+z}=dY\,,\ \text{et}$$

$$ydz-zdY=ydY.$$

Haec autem disponatur ita

$$z\left(\frac{dz}{z}-\frac{dY}{y}\right)=dY\,,$$

quae, facto

$$\frac{dY}{y}=\frac{ds}{s}\,,$$

mutabitur in

$$z\left(\frac{dz}{z}-\frac{ds}{s}\right)=dY\,,\ \text{seu}\ zdL\left(\frac{z}{s}\right)=sd\left(\frac{z}{s}\right)=dY.$$

Sumatur nunc $\frac{z}{s}=v$; proveniet

$$dv-\frac{dY}{s}=0\,,$$

aequatio cum variabilibus separatis, quia s data est per y.

156. Alterum artificium praebent quantitates indeterminatae in processu operationis ita determinandae ut, evanescentibus aliquot aequationis terminis, ii qui restant separationem admittant.

Exempla.

I.° Sit

$$dy + Xy\,dx = X_1 dx.$$

Fac $y = zs$, et adhibita substitutione orietur

$$z\,ds + s\,dz + Xzs\,dx = X_1 dx.$$

Jam si ponatur

$$s\,dz + Xzs\,dx = 0,\ \text{ideoque}\ \frac{dz}{z} + X\,dx = 0,$$

dabitur z per x ita, ut deletis binis terminis prodeat

$$z\,ds = X_1 dx,\ \text{seu}\ ds - \frac{X_1 dx}{z} = 0,$$

in qua variabiles inveniuntur separatae.

II.° Detur

$$(3y^2x - 3yx^2 - a^3 - y^3)dx + (a^3 + x^3)dy = 0;$$

et denotantibus α, μ quantitates determinandas ut commodum fuerit, z vero novam variabilem, ponatur $y = \alpha x^{\mu} + z$ ut ex data eliminentur y, dy: prodibit

$$\left.\begin{aligned}
&-\alpha^3 x^{3\mu}dx - 3\alpha^2 z x^{2\mu}dx - 3\alpha z^2 x^{\mu}dx - z^3 dx + x^3 dz + a^3 dz\\
&+3\alpha^2 x^{2\mu+1}dx + 6\alpha z x^{\mu+1}dx + 3z^2 x\,dx\\
&-3\alpha x^{\mu+2}dx - 3z x^2 dx\\
&-a^3 dx\\
&+\mu\alpha x^{\mu+2}dx\\
&+\mu\alpha a^3 x^{\mu-1}dx
\end{aligned}\right\} = 0$$

Fac ut tertius terminus evanescat, quod obtinebis assumendo $\mu=1$, $\alpha=1$: verum his valoribus adhibitis evanescent quoque secundus et primus; igitur aequatio traducetur ad

$$-z^2dx+x^3dz+a^3dz=0\text{, seu }\frac{dx}{a^3+x^3}-\frac{dz}{z^2}=0.$$

III.° Sit

$$(4a^2x^2+4a^2bx+4abxy+2ab^2y+b^2y^2)dx-ab^3dy=0.$$

Ad eliminandas y, dy utamur substitutione $y=\alpha x+z+\mu$; proveniet

$$\left.\begin{array}{l} 4a^2x^2dx+4a^2bxdx+2ab^2\mu dx+4abxzdx+2ab^2zdx+b^2z^2dx-ab^3dz \\ +4ab\alpha x^2dx+4ab\mu xdx+b^2\mu^2dx+2b^2\alpha xzdx+2b^2\mu zdx \\ +b^2\alpha^2x^2dx+2ab^2\alpha xdx-ab^3\alpha dx \\ \qquad +2b^2\alpha\mu xdx \end{array}\right\}=0$$

Determinetur α ita, ut evanescat primus terminus; fiat nimirum

$$b^2\alpha^2+4ab\alpha+4a^2=0\text{, unde }\alpha=-\frac{2a}{b};$$

hoc autem adhibito valore evanescent quoque secundus et quartus. Nunc si fit $\mu=-a$, evanescet quintus terminus, ac tertius evadet a^2b^2dx. Quibus positis, habebimus

$$a^2b^2dx+b^2z^2dx-ab^3dz=0\text{, seu }dx+\frac{abdz}{a^2+z^2}=0.$$

157. Interdum adhibendo utrumque artificium pervenitur ad separationem variabilium.

Exempla.

I.° Sit

$$(ay-x)dx-ydy=0\,;$$

et assumpta $y=\alpha x+z$, orietur

$$\left.\begin{array}{l}-xdx \quad +azdx-\alpha xdz-zdz\\ +a\alpha xdx-\alpha zdx\\ -\alpha^2 xdx\end{array}\right\}=0.$$

Ut primus terminus evanescat pone

$$\alpha^2-a\alpha+1=0\,,\ \text{unde}\ \ \alpha=\frac{a}{2}\pm\sqrt{\left[\frac{a^2}{4}-1\right]}\,;$$

restabit aequatio

$$(a-\alpha)zdx-\alpha xdz-zdz=0\,,$$

in qua possunt variabiles per primum artificium separari. Disponatur aequatio in hunc modum

$$(a-\alpha)\frac{dx}{x}-\alpha\frac{dz}{z}-\frac{dz}{x}=0\,,\ \text{sive}\ \ d\mathrm{L}\left(\frac{x^{a-\alpha}}{z^{\alpha}}\right)-\frac{dz}{x}=0\,;$$

et facto

$$\frac{x^{a-\alpha}}{z^{\alpha}}=s\,,\ \text{ut sit}\ x=s^{\frac{1}{a-\alpha}}\cdot z^{\frac{\alpha}{a-\alpha}}\,,$$

habebimus

$$\frac{ds}{s}-\frac{dz}{s^{\frac{1}{a-\alpha}}\cdot z^{\frac{\alpha}{a-\alpha}}}=0\,,\ \text{seu}\ s^{\frac{\alpha-a+1}{a-\alpha}}ds-z^{-\frac{\alpha}{a-\alpha}}dz=0$$

cum variabilibus separatis. Oportet autem ut α proveniat realis.

II.° Sit etiam

$$dy + Xydx = X_1 y^k dx,$$

quae disponatur ita

$$\frac{dy}{y^k} + X\frac{dx}{y^{k-1}} = X_1 dx,$$

$$\text{seu } -\frac{1}{k-1}d\left(\frac{1}{y^{k-1}}\right) + X\frac{dx}{y^{k-1}} = X_1 dx;$$

et facto $\frac{1}{y^{k-1}} = z$, prodibit

$$dz - (k-1)Xzdx = -(k-1)X_1 dx.$$

Haec habet eamdem formam, sub qua sese obtulit aequatio jam pertractata (90. I). Separabuntur ergo variabiles per secundum artificium.

III.° Detur

$$dy + (by^2 - ax^c)dx = 0 \ldots (o^{IV}).$$

Separatio nullo indiget artificio si $c=0$; nam habetur immediate

$$dx + \frac{dy}{by^2 - a} = 0.$$

Hoc primo casu observato, ut ad alios progrediar sumo $y = \alpha x^\mu + x^\nu z$, et facta substitutione obtineo

$$\left.\begin{array}{l} \nu x^{\nu-1} zdx + \mu\alpha x^{\mu-1} dx + bx^{2\nu} z^2 dx - ax^c dx + x^\nu dz \\ + 2b\alpha x^{\mu+\nu} zdx + b\alpha^2 x^{2\mu} dx \end{array}\right\} = 0$$

Primus ac secundus terminus evanescent si

$\nu-1=\mu+\nu$, $\nu+2b\alpha=0$, $\mu-1=2\mu$, $\mu\alpha+b\alpha^2=0$, quarum prima et tertia praebent $\mu=-1$, secunda et quarta $\alpha=\frac{1}{b}$, $\nu=-2$. Hinc

$$y=\frac{1}{bx}+\frac{z}{x^2},\ bz^2dx-ax^{c+4}dx+x^2dz=0\ldots(o^v).$$

Si $c=-4$, manifeste traducetur (o^v) ad

$$\frac{dx}{x^2}+\frac{dz}{bz^2-a}=0,$$

in qua variabiles exsistunt separatae.

Eadem (o^v) potest sic disponi

$$-\frac{dz}{z^2}+\frac{ax^{c+2}dx}{z^2}-\frac{bdx}{x^2}=0,$$

seu

$$d\frac{1}{z}+\frac{a}{(c+3)z^2}dx^{c+3}-\frac{bdx}{x^2}=0.$$

Positis $\frac{1}{z}=y_1$, $x^{c+3}=x_1$, ideoque

$$\frac{dx}{x^2}=\frac{1}{c+3}x_1^{-\frac{c+4}{c+3}}dx_1,\text{ exsurget}$$

$$dy_1+\frac{a}{c+3}y_1^2dx_1-\frac{b}{c+3}x_1^{-\frac{c+4}{c+3}}dx_1=0,$$

quae, factis $\frac{a}{c+3}=b_1$, $\frac{b}{c+3}=a_1$, $-\frac{c+4}{c+3}=c_1$, abit in

$$dy_1 + (b_1 y_1^2 - a_1 x_1^{c_1}) dx_1 = 0 \ldots (o^{VI}).$$

Haec autem similis omnino est aequationi (o^{IV}). Itaque adhibita

$$y_1 = \frac{1}{b_1 x_1} + \frac{z_1}{x_1^2},$$

fiet gradus ad

$$b_1 z_1^2 dx_1 - a_1 x_1^{c_1+4} dx_1 + x_1^2 dz_1 = 0 \ldots (o^{VII}),$$

in qua licebit separare z_1, x_1 si $c_1 = -4$. Disposita (o^{VII}) quemadmodum praestitimus quoad (o^{V}), sumptisque

$$\frac{1}{z_1} = y_2, \quad x_1^{c_1+3} = x_2, \quad \frac{a_1}{c_1+3} = b_2,$$

$$\frac{b_1}{c_1+3} = a_2, \quad -\frac{c_1+4}{c_1+3} = c_2,$$

eruetur ex (o^{VII})

$$dy_2 + (b_2 y_2^2 - a_2 x_2^{c_2}) dx_2 = 0 \ldots (o^{VIII});$$

t ex hac, ope substitutionis

$$y_2 = \frac{1}{b_2 x_2} + \frac{z_2}{x_2^2},$$

nova item deducetur aequatio inter variabiles x_2, z_2, quae poterit separari si $c_2 = -4$; atque ita porro.

Non pluribus opus est ut intelligamus variabiles x, y in (o^{IV}) fore separabiles quotiescumque una quaevis e quantitatibus

$$c, \; c_1 = -\frac{c+4}{c+3}, \; c_2 = -\frac{c_1+4}{c_1+3}, \; c_3 = -\frac{c_2+4}{c_2+3}, \ldots$$

erit $= -4$; sive, quod eodem redit, quotiescumque

c obtinebit unum quemvis e valoribus

$$-4\,,\ -\frac{8}{3}\,,\ -\frac{12}{5}\,,\ -\frac{16}{7}\,,\ -\frac{20}{9}\,,\ldots$$

qui possunt exprimi per formulam

$$c = -\frac{4n}{2n-1}\ \ldots\ \mathrm{o}(^{\mathrm{IX}})\,;$$

denotat n numerum integrum ac positivum.

Aequationem $(\mathrm{o}^{\mathrm{IV}})$ nunc disponamus ita

$$\frac{dy}{y^2} + bdx - \frac{ax^c dx}{y^2} = 0\,,\ \text{seu}$$

$$d\frac{1}{y} + \frac{a}{(c+1)y^2}dx^{c+1} - bdx = 0.$$

Ponantur $\frac{1}{y} = s\,,\ x^{c+1} = t$, ideoque

$dx = \frac{1}{c+1}t^{-\frac{c}{c+1}}dt$: adhibitis substitutionibus, factisque $\frac{a}{c+1} = \mathrm{B}\,,\ \frac{b}{c+1} = \mathrm{A}\,,\ -\frac{c}{c+1} = i$, orietur

$$ds + (\mathrm{B}s^2 - \mathrm{A}t^i)dt = 0\ \ldots\ (\mathrm{o}^{\mathrm{X}})\,;$$

quae cum prorsus sit similis ipsi $(\mathrm{o}^{\mathrm{IV}})$, easdem proinde ac $(\mathrm{o}^{\mathrm{IV}})$ transformationes admittet. Hinc pronum est concludere variabiles s, t in $(\mathrm{o}^{\mathrm{X}})$, et consequenter x, y in $(\mathrm{o}^{\mathrm{IV}})$ adhuc fore separabiles, si

$$i = -\frac{c}{c+1} = -\frac{4n}{2n-1}\,,\ \text{idest si}\ c = -\frac{4n}{2n+1}\ \ldots\ (\mathrm{o}^{\mathrm{XI}}).$$

158. Aequationes quaedam tametsi homogeneae non sint, tamen apte mutando exponentes convertuntur

in homogeneas ut possit (154) dein ad variabilium separationem transiri. Detur aequatio haud homogenea

$$\left.\begin{aligned}&(Ax^m y^n + Bx^{m'} y^{n'} + Hx^{m''} y^{n''} + \ldots)\,dx + \\ &(ax^k y^i + bx^{k'} y^{i'} + hx^{k''} y^{i''} + \ldots)\,dy = 0\end{aligned}\right\} (o^{\text{XII}}).$$

Facto $y = s^{\alpha}$, habebimus

$$(Ax^m s^{\alpha n} + Bx^{m'} s^{\alpha n'} + Hx^{m''} s^{\alpha n''} + \ldots)\,dx +$$

$$(ax^k s^{\alpha i} + bx^{k'} s^{\alpha i'} + hx^{k''} s^{\alpha i''} + \ldots)\,\alpha s^{\alpha - 1}\,ds = 0.$$

Haec autem exsistet homogenea, si

$$m + \alpha n = m' + \alpha n' = m'' + \alpha n'' = \ldots$$

$$= k + \alpha i + \alpha - 1 = k' + \alpha i' + \alpha - 1 = k'' + \alpha i'' + \alpha - 1 = \ldots;$$

igitur ubi fuerit

$$\frac{m' - m}{n - n'} = \frac{m'' - m}{n - n''} = \ldots = \frac{k - m - 1}{n - i - 1} = \frac{k' - m - 1}{n - i' - 1} = \ldots (o^{\text{XIII}}),$$

convertetur data aequatio in homogeneam accipiendo

$$y = s^{\frac{m' - m}{n - n'}}.$$

Exempla.

I.° Sit $(ay^3 x + by^2 x^{\frac{1}{2}})dx - cxdy = 0$; erunt $m = 1$, $n = 3$, $m' = \frac{1}{2}$, $n' = 2$, $k = 1$, $i = 0$, et $\dfrac{\frac{1}{2} - 1}{3 - 2} = -\frac{1}{2} = \dfrac{1 - 1 - 1}{3 - 1}$. Quare sumpta $y = s^{-\frac{1}{2}}$, vertetur aequatio in homogeneam $(axs^{-\frac{5}{2}} + bx^{\frac{1}{2}} s^{-1})dx + \dfrac{c}{2} xs^{-\frac{3}{2}} ds = 0.$

II.° In aequatione (157: II.°) $dy+(by^2-ax^c)dx=0$ habemus $m=0$, $n=2$, $m'=c$, $n'=0$, $k=0$, $i=0$; et $\frac{m'-m}{n-n'}=\frac{c}{2}$, $\frac{k-m-1}{n-i-1}=-1$. Hinc si $c=-2$, aequatio fiet homogenea sumendo $y=s^{-1}$.

159. Si fractionum (o^{XIII}) numeratores prodeunt omnes $=0$, ideoque $m'=m=m''=\ldots$, $k=m+1=k'=\ldots$, aequatio (o^{XII}) manifeste traducetur ad

$$\frac{dx}{x}+\frac{ay^i+by^{i'}+hy^{i''}+\ldots}{Ay^n+By^{n'}+Hy^{n''}+\ldots}dy=0.$$

Si denominatores sunt $=0$, ac proinde $n'=n=n''=\ldots$, $i=n-1=i'=\ldots$, aequatio (o^{XII}) fiet

$$\frac{Ax^m+Bx^{m'}+Hx^{m''}+\ldots}{ax^k+bx^{k'}+hx^{k''}+\ldots}dx+\frac{dy}{y}=0.$$

Si tam numeratores quam denominatores inveniuntur $=0$ ut fiant singulae fractiones $=\frac{0}{0}$, aequatio (o^{XII}) evadet

$$(A+B+H+\ldots)\frac{dx}{x}+(a+b+h+\ldots)\frac{dy}{y}=0.$$

Quod si nonnullae dumtaxat, e fractionibus (o^{XIII}) fierent $=\frac{0}{0}$, profecto ad reddendam homogeneam (o^{XII}) adhibendus esset valor ille exponentis α, quem exposcunt caeterae fractiones.

160. Non raro contingit quod ad reddendam aequationem homogeneam haud sufficiat mutare exponentes, sed oporteat adhibere alias substitutiones; pro quibus utiliter seligendis nulla suppetit regula. Sic v. gr. aequatio $(a\sqrt{[x^2+xy^2-ba]}-by)dx-bxdy=0$ nequit evadere homogenea per solam exponentium

mutationem ; at facto $xy - ba = z^2$, convertetur in homogeneam $a(x^2 + z^2)^{\frac{1}{2}}dx - 2bzdz = 0$.

Sic quoque data $(a+bx+cy)dx+(a_1+b_1x+c_1y)dy=0$, factisque $a+bx+cy=s$, $a_1+b_1x+c_1y=t$, et consequenter

$$x = \frac{c_1s - ct + a_1c - ac_1}{bc_1 - b_1c}, \; y = \frac{bt - b_1s + ab_1 - a_1b}{bc_1 - b_1c},$$

$$dx = \frac{c_1ds - cdt}{bc_1 - b_1c}, \quad dy = \frac{bdt - b_1ds}{bc_1 - b_1c},$$

prodibit aequatio homogenea

$$(c_1s - b_1t)ds + (bt - cs)dt = 0.$$

Bonum erit notare illud : si exsisteret $bc_1 - b_1c = 0$, forent $x = \infty$, $y = \infty$; at quoniam in casu aequatio data reciperet hanc formam

$$adx + (bx + cy)dx + a_1dy + \frac{c_1}{c}(bx + cy)dy = 0 ;$$

hinc facto

$$bx + cy = z, \text{ ideoque } dy = \frac{dz - bdx}{c},$$

proveniret

$$(a + z - \frac{a_1b}{c} - \frac{bc_1}{c^2}z)dx + (\frac{a_1}{c} + \frac{c_1}{c^2}z)dz = 0,$$

in qua variabiles immediate separantur.

161. Quod ad factorem φ (153) spectat, haud difficulter ostenditur ipsum semper exsistere : exhibeat enim V primum membrum aequationis (o), factisque $dx = x_1$, $dy = y_1$, ut sit

$$V = Mx_1 + Ny_1 = 0 \ldots (o^{XIV}),$$

ponatur

$$\varphi V = \perp d\mu \ldots (o^{xv}),$$

idest φV differentiale exactum. Quoniam (o^{xv}) importat sequentem

$$\varphi V = \frac{d\mu}{dx} x_1 + \frac{d\mu}{dy} y_1,$$

et consequenter etiam

$$\frac{d(\varphi V)}{dx} = \frac{d^2\mu}{dx^2} x_1 + \frac{d^2\mu}{dxdy} y_1 = d\left(\frac{d\mu}{dx}\right), \quad \frac{d(\varphi V)}{dx_1} = \frac{d\mu}{dx},$$

$$\frac{d(\varphi V)}{dy} = \frac{d^2\mu}{dxdy} x_1 + \frac{d^2\mu}{dy^2} y_1 = d\left(\frac{d\mu}{dy}\right), \quad \frac{d(\varphi V)}{dy_1} = \frac{d\mu}{dy};$$

iccirco in ea qua sumus hypothesi binae

$$\frac{d(\varphi V)}{dx} - d\left(\frac{d(\varphi V)}{dx_1}\right) = 0, \quad \frac{d(\varphi V)}{dy} - d\left(\frac{d(\varphi V)}{dy_1}\right) = 0$$

seu

$$\left.\begin{aligned} \varphi\frac{dV}{dx} - \frac{dV}{dx_1}d\varphi - \varphi d\left(\frac{dV}{dx_1}\right) &= -V\frac{d\varphi}{dx}, \\ \varphi\frac{dV}{dy} - \frac{dV}{dy_1}d\varphi - \varphi d\left(\frac{dV}{dy_1}\right) &= -V\frac{d\varphi}{dy} \end{aligned}\right\} (o^{xvi})$$

necesse est proveniant idemticae; et cum habeamus $V=0$, hinc binae quoque

$$\left.\begin{aligned} \varphi\frac{dV}{dx} - \frac{dV}{dx_1}d\varphi - \varphi d\left(\frac{dV}{dx_1}\right) &= 0, \\ \varphi\frac{dV}{dy} - \frac{dV}{dy_1}d\varphi - \varphi d\left(\frac{dV}{dy_1}\right) &= 0 \end{aligned}\right\} (o^{xvii})$$

debent esse idemticae quum in ipsis substituitur valor y_1 ex $V=0$; atque idipsum dicendum de

$$\frac{dV}{dx}\cdot\frac{dV}{dy_1}-\frac{dV}{dy_1}d\left(\frac{dV}{dx_1}\right)+\frac{dV}{dx_1}d\left(\frac{dV}{dy_1}\right)-\frac{dV}{dy}\cdot\frac{dV}{dx_1}=0..(o^{\mathrm{xviii}}),$$

quae obtinetur eliminando $\frac{d\varphi}{\varphi}$ ex (0^{xvii}) : vicissim si (o^{xviii}) prodit idemtica, haud dubie valebit (o^{xv}), et consequenter aequatio (o) integrabilis ; secus neque binae (o^{xvi}) neque icciroo binae (o^{xvii}) provenirent idemticae ; nullus videlicet foret valor φ, nullusque per consequens valor $\frac{d\varphi}{\varphi}$ duabus (o^{xvii}) satisfaciens, et aequatio (o^{xviii}) non prodiret idemtica. Jam vero ex (o^{xiv}) habemus

$$\frac{dV}{dx}=\frac{dM}{dx}x_1+\frac{dN}{dx}y_1,\ \frac{dV}{dy_1}=N,\ \frac{dV}{dx_1}=M,$$
$$\frac{dV}{dy}=\frac{dM}{dy}x_1+\frac{dN}{dy}y_1,$$

quarum ope traducitur (o^{xviii}) ad

$$(Mx_1+Ny_1)\left(\frac{dN}{dx}-\frac{dM}{dy}\right)=0;$$

quae cum revera ob (o^{xiv}) sit idemtica, licebit inferre factorem φ semper exsistere, et aequationem (o) semper derivari posse ex aliqua primitiva aequatione inter x, y.

Haec notentur. 1.° ad inveniendum φ, habemus quidem ex (o''')

$$M\frac{d\varphi}{dy}-N\frac{d\varphi}{dx}+\left(\frac{dM}{dy}-\frac{dN}{dx}\right)\varphi=0\ .\ .\ .\ (o^{\mathrm{xix}});$$

verum differentialis, partialisque (52 : 53) aequatio (o^{xix}) difficilius integratur quam data (o) : quare hic quoque nihil ferme superest aliud quo juvemur nisi

usus et industria. 2.° caeterum si cognoscitur aliquis valor φ_1 per quem convertatur (o) in differentiale exactum, poterunt inde innumeri alii valores φ derivari ipsam (o) pariter constituentes differentiale exactum. Sit namque

$$\varphi_1 M dx + \varphi_1 N dy = d\mu :$$

certe aequatio

$$\varphi_1 \chi(\mu) M dx + \varphi_1 \chi(\mu) N dy = \chi(\mu) d\mu$$

erit idemtica; secundum vero membrum est differentiale exactum, igitur et primum; ideoque cognito φ_1 apto ad constituendam (o) integrabilem, eruentur inde innumeri alii valores

$$\varphi = \varphi_1 \chi(\mu),$$

per quos fiet (o) similiter integrabilis. 3.° si in (o) functiones M et N sunt homogeneae, erit

$$\varphi_1 = \frac{1}{Mx + Ny}.$$

Nam

$$1.^a \; \frac{d(\varphi_1 M)}{dy} = \frac{1}{(Mx+Ny)^2}\left(Ny\frac{dM}{dy} - My\frac{dN}{dy} - MN\right),$$

$$2.^a \; \frac{d(\varphi_1 N)}{dx} = \frac{1}{(Mx+Ny)^2}\left(Mx\frac{dN}{dx} - Nx\frac{dM}{dx} - MN\right);$$

et in ea qua sumus hypothesi (64)

$$3.^a \; kM = x\frac{dM}{dx} + y\frac{dM}{dy}, \quad 4.^a \; kN = x\frac{dN}{dx} + y\frac{dN}{dy};$$

designat k gradum functionum M, N. Jam in 1.^{a} substituantur valores $y\frac{dM}{dy}, y\frac{dN}{dy}$ ex 3.^{a} et 4.^{a}, animusque attendatur ad 2^{am}; prodibit (o'''), ideoque

et caet. . . . 4.° pervenitur aliquando ad factorem φ_1 partiendo aequationem (o) in binas partes ita, ut respectivus singularum factor possit facile cognosci. Sint

$$M_1dx+N_1dy\ ,\ M_2dx+N_2dy$$

partes illae ut habeamus

$$M_1dx+N_1dy+M_2dx+N_2dy=0\ ,$$

sintque f_1, f_2 factores per quos eae convertuntur in differentialia exacta, nimirum

$$f_1M_1dx+f_1N_1dy=d\mu_1\ ,\ f_2M_2dx+f_2N_2dy=d\mu_2\ ;$$

expressionum $f_1\chi_1(\mu_1)$, $f_2\chi_2(\mu_2)$ altera repraesentabit (1.°) factores omnes qui partem primam respiciunt, altera factores omnes qui secundam: nunc si functionum χ_1, χ_2 forma determinatur ita ut expressiones illae fiant idemticae, erit in promptu φ_1. Rem declaramus exemplo: detur

$$aydx+bxdy+x^my^n(hydx+kxdy)=0$$

ut sint $M_1=ay$, $N_1=bx$, $M_2=hx^my^{n+1}$, $N_2=kx^{m+1}y^n$. Attendenti patebit fore

$$f_1=\frac{1}{xy}\ ,\ f_2=\frac{1}{x^{m+1}y^{n+1}}\ ,$$

$$d\mu_1=dL(x^ay^b)\ ,\ d\mu_2=dL(x^hy^k).$$

Expressiones

$$\frac{1}{xy}\chi_1(x^ay^b)\ ,\ \frac{1}{x^{m+1}y^{n+1}}\chi_2(x^hy^k)$$

fient idemticae si, positis

$$\chi_1(x^ay^b)=x^{\alpha a}y^{\alpha b}\ ,\ \chi_2(x^hy^k)=x^{\alpha_1h}y^{\alpha_1k}\ ,$$

habeantur

$$\alpha a-1=\alpha_1h-m-1\ ,\ \alpha b-1=\alpha_1k-n-1\ .$$

Hinc

$$\alpha = \frac{nh-mk}{ak-bh}, \quad \alpha_1 = \frac{an-mb}{ak-bh};$$

et

$$\varphi_1 = \frac{1}{xy}(x^a y^b)^{\frac{nh-km}{ak-bh}}.$$

5.° aequatio (o^{xix}) scribatur ita

$$\frac{1}{\varphi}\left(\frac{d\varphi}{dx} - \frac{M}{N}\cdot\frac{d\varphi}{dy}\right) = \frac{1}{N}\left(\frac{dM}{dy} - \frac{dN}{dx}\right):$$

si secundum membrum prodit functio X solius x, tale erit et primum; quod cum certe assequimur ponendo φ functionem solius x, icirco in casu

$$\frac{d\varphi}{\varphi} = \frac{1}{N}\left(\frac{dM}{dy} - \frac{dN}{dx}\right)dx = Xdx;$$

unde

$$\varphi = e^{\int Xdx}:$$

simili modo invenietur φ, si in

$$\frac{1}{M}\left(\frac{dN}{dx} - \frac{dM}{dy}\right)$$

ingreditur sola y.

162. Differentialia dx, dy haud raro sese offerunt elata ad potestates altiores prima: universalis ejusmodi aequationum forma est

$$dy^n + Pdy^{n-1}dx + Qdy^{n-2}dx^2 + \ldots + Tdydx^{n-1} + Vdx^n = 0,$$

seu

$$\left(\frac{dy}{dx}\right)^n + P\left(\frac{dy}{dx}\right)^{n-1} + Q\left(\frac{dy}{dx}\right)^{n-2} + \ldots + T\left(\frac{dy}{dx}\right) + V = 0 \ldots (o^{xx}).$$

Ponatur (o^{xx}) esse homogenea quoad x, y: factis $y = xz$, $xdz = tdx$ ut sit $dy = (z+t)dx$, traducetur (o^{xx}) ad

$(z+t)^n + P_1(z+t)^{n-1} + Q_1(z+t)^{n-2} + \ldots + T_1(z+t) + V_1 = 0 \ldots (o^{xxi})$,

in qua P_1, Q_1, ... V_1 designant functiones unius z; datur ergo t per z et vicissim, ideoque aequatio primi ordinis primique gradus

$$xdz = tdx \text{ seu } \frac{dx}{x} = \frac{dz}{t}$$

praebebit x expressam per z; unde et caet. .... Sit v. gr.

$$\left(\frac{dy}{dx}\right)^3 - \tfrac{1}{2}\left(\frac{dy}{dx}\right)^2 + \frac{x}{y}\left(\frac{dy}{dx}\right) - \frac{x}{2y} = 0.$$

Erunt $n=3$, $P_1 = -\frac{1}{2}$, $Q_1 = \frac{1}{z}$, $V_1 = -\frac{1}{2z}$; quare

$$(z+t)^3 - \tfrac{1}{2}(z+t)^2 + \frac{1}{z}(z+t) - \frac{1}{2z} = 0$$

quae sic potest scribi

$$(2(z+t)-1)\left(\frac{1+z(z+t)^2}{2z}\right) = 0\;;$$

unde

$$2(z+t)-1=0\;,\; 1+z(z+t)^2=0.$$

Harum prima suppeditat

$$t = \tfrac{1}{2} - z\;,\text{ secunda } t = -z \pm \left(-\frac{1}{z}\right)^{\frac{1}{2}};$$

hinc

$$\frac{dx}{x} = \frac{dz}{\frac{1}{2}-z}\;,\; \frac{dx}{x} = \frac{dz}{-z+\left(-\frac{1}{z}\right)^{\frac{1}{2}}},\; \frac{dx}{x} = -\frac{dz}{z+\left(-\frac{1}{z}\right)^{\frac{1}{2}}}.$$

Jam si prius integrentur aequationes istae, ac dein adhibeatur $\frac{y}{x}$ pro z, exsurgent terna integralia aequationis propositae. Eadem methodo expediuntur aequationes illae, quae licet non homogeneae, possunt tamen (158) ad homogeneitatem perduci.

Si P, Q, ... V sunt quantitates constantes, talis quoque erit $\frac{dy}{dx}$; et facto $\frac{dy}{dx}=a$, ut sint

$$a^n+Pa^{n-1}+Qa^{n-2}+\ldots+Ta+V=0,\ y=ax+C,\ a=\frac{y-C}{x},$$

exsistet

$$\left(\frac{y-C}{x}\right)^n+P\left(\frac{y-C}{x}\right)^{n-1}+Q\left(\frac{y-C}{x}\right)^{n-2}+\ldots+T\left(\frac{y-C}{x}\right)+V=0$$

integrale propositae aequationis.

163. Utcumque se habeat (o^{xx}), si ea potest resolvi quoad $\frac{dy}{dx}$, sintque p, q, ... t, v radices inde prodeuntes, habebimus n aequationes differentiales primi ordinis, primique gradus

$$\frac{dy}{dx}-p=0,\ \frac{dy}{dx}-q=0,\ \ldots\ \frac{dy}{dx}-t=0,\ \frac{dy}{dx}-v=0,$$

quibus pertractatis juxta superiores regulas ut obtineantur n integralia, spectabunt singula ad (o^{xx}).

Exempla.

I.° Sit $dy^2-axdx^2=0$, ex cujus resolutione assequimur

$$dy+dx\sqrt{ax}=0,\ dy-dx\sqrt{ax}=0:$$

prodibunt bina integralia

$$y+\frac{2}{3}a^{\frac{1}{2}}x^{\frac{3}{2}}-C=0\,,\ y-\frac{2}{3}a^{\frac{1}{2}}x^{\frac{3}{2}}-C_1=0.$$

II.° Detur

$$\left(\frac{dy}{dx}\right)^2+2\frac{y}{x}\cdot\frac{dy}{dx}=1.$$

Facta resolutione, provenient binae homogeneae

$$\frac{dy}{dx}=-\frac{y}{x}+\sqrt{\left[\frac{y^2}{x^2}+1\right]},\ \frac{dy}{dx}=-\frac{y}{x}-\sqrt{\left[\frac{y^2}{x^2}+1\right]},$$

quibus aptari poterit methodus jam tradita (154. 2.°).

Caeterum cum generalis aequationum resolutio in potestate nostra non sit, cumque etiam quando eas licet resolvere, expressiones radicum saepe proveniant haud parum intricatae, hinc est quod plerumque ad peculiaria artificia oportet confugere. Sit

$$x=yR+aS\ \ldots\ (0^{XXI}).$$

in qua R, S ponuntur datae per $\frac{dy}{dx}$ et constantes utcumque: sumpto $dx=zdy$, adhibitisque substitutionibus, disparebunt dx, dy ex R, S, et aequatio recipiet hanc formam

$$\int zdy=yR_1+aS_1\,,$$

in qua R_1, S_1 dantur per z et constantes. Jam si aequatio haec differentietur, proveniet

$$zdy=R_1dy+ydR_1+adS_1\ \ldots\ (0^{XXII})\,;$$

quae quum scribi potest in hunc modum

$$dy-\frac{ydR_1}{z-R_1}=\frac{adS_1}{z-R_1}\,,$$

profecto erit separabilis (156. I.°) : eruetur itaque integrale aequationis (o^{XXI}) ex (o^{XXII}) et ex $dx = zdy$. Primum exemplum praebeat aequatio

$$x = \frac{ydx}{dy} + \frac{adx^2}{dy^2}:$$

erunt $R_1 = z$, $S_1 = z^2$, et $zdy = zdy + ydz + 2azdz$, seu $(y + 2az)dz = 0$; hinc binae

$$dz = 0,\ y + 2az = 0.$$

Prima suppeditat $z = C$, ideoque $dx = Cdy$, $x = Cy + C_1$: in hanc formulam duae ingrediuntur constantes; sed cum in data aequatione non inveniantur nisi differentialia primi ordinis, unica dumtaxat constans videtur locum habere posse : quare altera erit per alteram determinanda. Substituantur itaque in aequatione data valores x, dx; prodibit $Cy + C_1 = Cy + aC^2$, et $C_1 = aC^2$; unde verum integrale

$$y = Cy + aC^2.$$

Quod ad secundam spectat, habemus

$$z = -\frac{y}{2a}, \text{ et consequenter } dx = -\frac{ydy}{2a},\ x = C - \frac{y^2}{4a}:$$

est $C = 0$; nam substitutis valoribus x, dx in aequatione data, emerget

$$C - \frac{y^2}{4a} = -\frac{y^2}{2a} + \frac{y^2}{4a} = -\frac{y^2}{4a}.$$

Alterum exemplum desumo ab

$$x = -\frac{ydx}{dy} + \frac{adx^2}{dy^2} - b:$$

habemus

$$R = -\frac{dx}{dy},\ S = \frac{dx^2}{dy^2} - \frac{b}{a},$$

unde $R_1 = -z$, $S_1 = z^2 - \frac{b}{a}$; ideoque

$2zdy + ydz = 2azdz$, seu $2\sqrt{z}.\sqrt{z}\,dy + ydz = 2a\sqrt{z}.\sqrt{z}\,dz$, et $\sqrt{z}\,dy + \frac{ydz}{2\sqrt{z}} = a\sqrt{z}dz$, cujus

integrale $y\sqrt{z} = \frac{2}{3}az^{\frac{3}{2}} + C$: hinc

$$y = \frac{2}{3}az + \frac{C}{z^{\frac{1}{2}}}.$$

Inventa y per z, inquiramus in x. Quoniam $zy - \int ydz = \int zdy = x$, erit igitur

$$x = \frac{2}{3}az^2 + Cz^{\frac{1}{2}} - \frac{2}{3}a\int zdz - C\int\frac{dz}{z^{\frac{1}{2}}},\ \text{idest}$$

$$x = \frac{1}{3}az^2 - Cz^{\frac{1}{2}} + C_1:$$

alteram et constantibus C et C_1 determinemus. In aequatione data fiat $dx = zdy$, et pro x, y adhibeantur valores modo inventi; exsurget

$$\frac{1}{3}az^2 - Cz^{\frac{1}{2}} + C_1 = -z\left(\frac{2}{3}az + \frac{C}{z^{\frac{1}{2}}}\right) + az^2 - b;$$

unde $C_1 = -b$: habitis autem x et y per z, si haec eliminetur, prodibit aequatio inter x, y, a, b et arbitrariam C.

164. In aequatione

$$x=\mu$$

ponatur μ dari per y, $\frac{dy}{dx}$ et constantes utcumque. Facto ut supra $dx=zdy$, erit

$$\int zdy=\mu_1,$$

in qua μ_1 dabitur tantummodo per y, z et constantes: acceptis differentialibus,

$$zdy=d\mu_1;$$

et quoniam $d\mu_1$ datur per y, z, dy, dz et constantes ita, ut differentialia unicam dumtaxat obtineant dimensionem, iccirco postrema aequatio tractari poterit juxta notas. Sit v. gr. 1.°

$$x=\frac{ydx}{dy}+y^n u;$$

datur u quomodocumque per $\frac{dy}{dx}$ et constantes: erit $\mu_1=yz+y^n u_1$, et $ydz+nu_1 y^{n-1}dy+y^n du_1=0$; coalescit u_1 e z et constantibus. Fiat $y^n u_1=v$, ideoque

$$ydz=\left(\frac{v}{u_1}\right)^{\frac{1}{n}}dz,\ nu_1 y^{n-1}dy=\frac{u_1 dv-vdu_1}{u_1},$$

$$y^n du_1=\frac{vdu_1}{u_1};$$

prodibit

$$\left(\frac{v}{u_1}\right)^{\frac{1}{n}}dz+dv=0,\ \text{seu}\ \frac{dz}{u_1^{\frac{1}{n}}}+\frac{dv}{v^{\frac{1}{n}}}=0$$

cum variabilibus separatis. Sit 2.°

$$x = \frac{y dx}{dy} + \frac{a y^2 dx^2}{dy^2} + \frac{b y^3 dx^3}{dy^3} + \ldots$$

Erit

$$\mu_1 = yz + ay^2 z^2 + by^3 z^3 + \ldots,$$

ideoque

$$y dz + 2ayz^2 dy + 2ay^2 z dz + 3by^2 z^3 dy + 3by^3 z^2 dz + \ldots = 0.$$

Facto $yz = v$, et consequenter

$$y = \frac{v}{z}, \quad dy = \frac{z dv - v dz}{z^2},$$

habebimus

$$\frac{dz}{z} + 2a dv + 3bv dv + \ldots = 0, \text{ et } \mathrm{L}(z) + 2av + \frac{3}{2} bv^2 + \ldots = \mathrm{G}.$$

165. Juvat hic aequationem

$$\left.\begin{array}{l} \dfrac{dx}{\sqrt{[a_1 x^4 + a_2 x^3 + a_3 x^2 + a_4 x + a_5]}} = \\ \dfrac{dy}{\sqrt{[a_1 y^4 + a_2 y^3 + a_3 y^2 + a_4 y + a_5]}} \end{array}\right\} \quad (0^{\text{XXIII}})$$

pertractare, quae habet variabiles separatas, quaeque suo gaudet integrali algebraico, licet ejus membra seorsim sumpta neque algebraice, neque per logarithmos, neque per circulares arcus queant accurate integrari. Fac

$$\left.\begin{array}{r} \sqrt{[a_1 x^4 + a_2 x^3 + a_3 x^2 + a_4 x + a_5]} = \sqrt{\chi} \\ \sqrt{[a_1 y^4 + a_2 y^3 + a_3 y^2 + a_4 y + a_5]} = \sqrt{v}, \\ dx = du\sqrt{\chi}; \end{array}\right\} \quad (0^{\text{XXIV}})$$

habebis

$$\chi = \frac{dx^2}{du^2},\ \nu = \frac{dy^2}{du^2} \ldots (0^{xxv});$$

unde

$$\frac{dx^2-dy^2}{du^2} = \chi - \nu = a_1(x^4-y^4)+a_2(x^3-y^3)+a_3(x^2-y^2)+a_4(x-y),$$

quae (positis $x - y = v$, $x + y = z$, ideoque $dx - dy = dv$, $dx + dy = dz$) vertitur in

$$\frac{dvdz}{vdu^2} = \frac{a_1}{2}(z^3 + v^2 z) + \frac{a_2}{4}(3z^2 + v^2) + a_3 z + a_4.$$

Jam vero ex (0^{xxiv})

$$d\chi = (4a_1x^3+3a_2x^2+2a_3x+a_4)dx,$$

$$d\nu = (4a_1y^3+3a_2y^2+2a_3y+a_4)dy;$$

ex (0^{xxv}), habita du pro invariabili,

$$d\chi = \frac{2dxd^2x}{du^2},\ d\nu = \frac{2dyd^2y}{du^2},$$

et consequenter

$$\frac{2d^2x}{du^2} = 4a_1x^3 + 3a_2x^2 + 2a_3x + a_4,$$

$$\frac{2d^2y}{du^2} = 4a_1y^3 + 3a_2y^2 + 2a_3y + a_4,$$

ex quarum summa

$$\frac{d^2x + d^2y}{du^2} = \frac{d(dx+dy)}{du^2} = 2a_1(x^3+y^3) +$$

$$\frac{3a_2}{2}(x^2+y^2) + a_3(x+y) + a_4,$$

seu

$$\frac{d^2z}{du^2}=\frac{a_1}{2}(z^3+3zv^2)+\frac{3a_2}{4}(z^2+v^2)+a_3z+a_4:$$

igitur

$$\frac{d^2z}{du^2}-\frac{dvdz}{vdu^2}=a_1zv^2+\frac{a_2}{2}v^2.$$

Hanc multiplica per $\frac{2dz}{v^2}$; obveniet

$$\frac{1}{du^2}\left(\frac{2dzd^2z}{v^2}-\frac{2dvdz^2}{v^3}\right)=\frac{1}{du^2}d\left(\frac{dz^2}{v^2}\right)=2a_1zdz+a_2dz:$$

quare

$$\frac{dz}{du}=v\sqrt{[a_1z^2+a_2z+C]};$$

et restitutis valoribus

$$z=x+y,\ v=x-y,\ \frac{dz}{du}=\frac{dx}{du}+\frac{dy}{du}=\sqrt{X}+\sqrt{Y}=\text{ et}$$

caet. . . . , erit

$$\left.\begin{array}{l}\sqrt{(a_1x^4+a_2x^3+a_3x^2+a_4x+a_5)}+\\ \sqrt{(a_1y^4+a_2y^3+a_3y^2+a_4y+a_5)}=\\ (x-y)\sqrt{[a_1(x+y)^2+a_2(x+y)+C]}\end{array}\right\}(0^{\text{xxvi}})$$

integrale aequationis (0^{xxiii}).

Datis

$$\left.\begin{array}{c}\dfrac{dx}{\sqrt{(A_1x^6+A_2x^4+A_3x^2+A_4)}}=\\ \dfrac{dy}{\sqrt{(A_1y^6+A_2y^4+A_3y^2+A_4)}}\end{array}\right\}(0^{\text{xxvii}}),$$

$$\frac{dx}{x\sqrt{(A_1x^{2n}+A_2x^n+A_3)}}=\frac{dy}{y\sqrt{(A_1y^{2n}+A_2y^n+A_3)}}\ldots(0^{\text{xxviii}}),$$

fiat quoad (o^{xxvii})

$$x^2 = X\,,\ y^2 = Y\,,\ \text{unde}\ dx = \frac{dX}{2\sqrt{X}}\,,\ dy = \frac{dY}{2\sqrt{Y}}\,;$$

quoad (o^{xxviii})

$$x^n = X\,,\ y^n = Y\,,\ \text{unde}$$

$$dx = \frac{1}{n} X^{\frac{1-n}{n}} dX\,,\ dy = \frac{1}{n} Y^{\frac{1-n}{n}} dY.$$

Immutabitur (o^{xxvii}) in

$$\frac{dX}{\sqrt{(A_1X^4 + A_2X^3 + A_3X^2 + A_4X)}} = \frac{dY}{\sqrt{(A_1Y^4 + A_2Y^3 + A_3Y^2 + A_4Y)}}$$

ejusdem formae ac (o^{xxiii}), facta $a_5 = 0$.

Immutabitur (o^{xxviii}) in

$$\frac{dX}{X\sqrt{(A_1X^2 + A_2X + A_3)}} = \frac{dY}{Y\sqrt{(A_1Y^2 + A_2Y + A_3)}}$$

ejusdem formae ac (o^{xxiii}), factis $a_4 = 0$, $a_5 = 0$. Hinc ope formulae (o^{xxvi}) absque ullo negotio integrantur etiam (o^{xxvii}), (o^{xxviii}).

166. Si in integrali aequationis

$$f(x\,,\ y\,,\ \frac{dy}{dx}) = 0 \ldots (o_1)$$

tribuuntur arbitrariae C alii atque alii constantes valores, palam est alios pariter atque alios proventuros integralis *peculiares valores*, qui aequationi (o_1) satisfacient. Verum inter variabiles x, y sunt certae relationes, quae licet aequationi (o_1) satisfaciant, nul-

lo tamen pacto ex praedictis arbitrariae C determinatis valoribus exsurgunt, uti videre est in (163)

$$x=\frac{ydx}{dy}+\frac{adx^2}{dy^2}:$$

huic enim satisfacit relatio

$$x=-\frac{y^2}{4a},$$

quam profecto nullus ex iis valoribus constantibus suppeditabit, qui tribui possunt arbitrariae C in integrali

$$x=Cy+aC^2;$$

ejusmodi relationes dicuntur *solutiones particulares*.

167. Integrale aequationis (o_1) exhibeatur per

$$F(x, y, C)=0 \ldots (o_2):$$

exsistet

$$F'_x+F'_y.\frac{dy}{dx}=0 \ldots (o_3),$$

ex qua si eruis C ut ejus valorem substituas in (o_2), habebis (o_1). Sic data v. gr.

$$(2\sqrt{xy}+x)dy+ydx=0,$$

divide prius per $2\sqrt{xy}$, dein integra; prodibit

$$y+\sqrt{xy}=C, \text{ seu } y^2-2Cy+C^2-xy=0:$$

haec differentiata suppeditat

$$(2y-x-2C)dy-ydx=0, \text{ unde } C=y-\frac{x}{2}-\frac{ydx}{2dy};$$

et consequenter

$$y+\sqrt{xy}=y-\frac{x}{2}-\frac{ydx}{2dy}, \text{ seu } (2\sqrt{xy}+x)dy+ydx=0,$$

eadem prorsus ac data. Nunc si spectatur C ut functio quantitatis x, aequatio (o_2) differentiata quoad x, y, C praebebit

$$F'_x + F'_y \cdot \frac{dy}{dx} + F'_c \cdot \frac{dC}{dx} = 0 \ldots (o_4).$$

Haec autem residit in (o_3), ubi ponatur

$$F'_c = 0 \ldots (o_5):$$

in ea igitur qua sumus hypothesi proveniet semper (o_1) sive adhibeatur (o_3), sive (o_4) ad eliminandam C. Inde colligitur, si, resoluta (o_5) quoad C, exsurgat $C = \theta$, colligitur inquam fore

$$F(x, y, \theta) = 0 \ldots (o_6)$$

particularem solutionem aequationis (o_1); modo tamen θ non exsistat vel constans, vel talis ut ejus in (o_2) substitutio eo redeat ac si constans peculiarisque valor arbitrariae C substitueretur; tunc enim (o_6) nihil esset aliud nisi peculiaris valor ipsius (o_2).

Exempla.

I.° Resumatur (163) aequatio

$$x = \frac{ydx}{dy} + \frac{adx^2}{dy^2},$$

cujus integrale

$$x = Cy + aC^2:$$

erit $(o_5) = 2aC + y = 0$, ideoque $C = -\frac{y}{2a}$. Hinc particularis solutio $x = -\frac{y^2}{4a}$.

II.° Proponatur aequatio

$$xdx + ydy = dy \cdot \sqrt{(x^2 + y^2 - a^2)},$$

cujus integrale

$\sqrt{(x^2+y^2-a^2)}=y+C$, seu $x^2-2yC-a^2-C^2=0$:

erit $(o_s)=-2y-2C=0$, et consequenter $C=-y$. Hinc particularis solutio $x^2+y^2-a^2=0$.

III.° Sit

$$dy=\frac{xdx}{\sqrt{[1-x^4]}},$$

quae, facto $x^2=2z$, recipit hanc formam

$$dy=\tfrac{1}{2}\frac{d2z}{\sqrt{[1-(2z)^2]}};$$

ideoque ejus integrale:

$$2y+C=arc(sin=2z), \text{ seu } x^2=sin(2y+C):$$

erit igitur $(o_s)=cos(2y+C)=0$, ex qua $C=\frac{\pi}{2}-2y$; unde solutio particularis $x^2=1$.

IV.° Detur aequatio

$$(x^2+y^2-a)(y-2Cy)+(x^2-a)C^2=0:$$

erit

$(o_5)=-y(x^2+y^2-a)+(x^2-a)C=0$, ex qua $C=\frac{y(x^2+y^2-a)}{x^2-a}$; et consequenter

$$(o_s)=\frac{y^h(x^2+y^2-a)}{x^2-a}=0; \text{ hinc vero } x^2+y^2-a=0.$$

At quia haec resultat quoque ex peculiari valore $C=0$ substituto in aequatione data, iccirco non erit particularis solutio, sed peculiaris valor.

168. Aequationem (o_s) ponamus resolvi quoad C ut

habeatur $C = \mu(x, y)$; erit idemtice $F(x, y, \mu) = 0$; ideoque

$$F'_x + F'_\mu \cdot \mu'_x = 0, \quad F'_y + F'_\mu \cdot \mu'_y = 0, \text{ unde}$$

$$\mu'_x = -\frac{F'_x}{F'_\mu}, \quad \mu'_y = -\frac{F'_y}{F'_\mu}.$$

Atqui solutio particularis importat (167) $F'_\mu = 0$; importabit ergo et

$$\mu'_x = \infty, \quad \mu'_y = \infty:$$

hinc alia methodus inveniendi solutiones particulares. Ut rem declaremus exemplo, resumatur (167 : II.°) aequatio

$$x^2 - 2Cy - C^2 - a^2 = 0, \text{ ex qua}$$

$$C = -y + \sqrt{[x^2 + y^2 - a^2]}:$$

erunt

$$\mu'_x = \frac{x}{\sqrt{[x^2+y^2-a^2]}} = \infty, \quad \mu'_y = -1 + \frac{y}{\sqrt{[x^2+y^2-a^2]}} = \infty,$$

quae praebent solutionem particularem $x^2+y^2-a^2=0$.

169. Aequatio (o_1) ponatur traduci ad hanc formam

$$\frac{dy}{dx} + \omega(x, y) = 0 \ldots (o_7);$$

et resoluta (o_3) quoad C, emergat

$$C = \nu(x, y, \frac{dy}{dx}).$$

Quoniam (167), adhibita substitutione in (o_2), prodit

$$F(x, y, \nu(x, y, \frac{dy}{dx})) = 0,$$

eadem prorsus ac (o_7), iccirco exsistet idemtice

$$F(x, y, \nu(x, y, -\omega)) = 0 ;$$

proinde

$$F'_x + (\nu'_x - \nu'_\omega . \omega'_x)F'_\nu = 0 , \quad F'_y + (\nu'_y - \nu'_\omega . \omega'_y)F'_\nu = 0 ,$$

ex quibus

$$\omega'_x = \frac{F'_x + F'_\nu . \nu'_x}{F'_\nu . \nu'_\omega} , \quad \omega'_y = \frac{F'_y + F'_\nu . \nu'_y}{F'_\nu . \nu'_\omega} :$$

atqui solutio particularis importat $F'_\nu = 0$; importabit ergo et

$$\omega'_x = \infty , \quad \omega'_y = \infty :$$

hinc methodus eliciendi solutiones particulares ab ipsa aequatione differentiali quin cognoscatur ejus integrale. Sumentes iterum aequationem

$$xdx + ydy = dy.\sqrt{[x^2 + y^2 - a^2]} , \text{ seu}$$

$$\frac{dy}{dx} - \frac{x}{-y + \sqrt{[x^2 + y^2 - a^2]}} = 0 ,$$

habemus

$$\omega(x , y) = - \frac{x}{-y + \sqrt{[x^2 + y^2 - a^2]}} ;$$

propterea

$$\omega'_x = - \frac{1}{-y + \sqrt{[x^2 + y^2 - a^2]}} +$$

$$\frac{x^2}{(x^2 + y^2 - a^2)^{\frac{1}{2}} (-y + \sqrt{[x^2 + y^2 - a^2]})} = \infty ,$$

$$\omega'_y = - \frac{x}{(x^2 + y^2 - a^2)^{\frac{1}{2}} (-y + \sqrt{[x^2 + y^2 - a^2]})} = \infty ;$$

quibus satisfit per binas

$$x^2+y^2-a^2=0\ ,\ x^2-a^2=0\ .$$

Jamvero istarum prima suppeditat $xdx+ydy=0$, secunda $xdx=0$; per alteram igitur data aequatio traducitur ad idemticam $0=0$, per alteram ad $ydy=ydy$ similiter idemticam : sequitur binas aequationi datae fore solutiones particulares. Sed haec hactenus; nunc de aequationibus differentialibus, quae ternas complectuntur variabiles.

170. Proponatur aequatio

$$Mdx+Ndy+Pdz=0\ldots(o_8)\ ;$$

designant M, N, P functiones variabilium x, y, z. Vide utrum expleantur conditiones

$$\frac{dM}{dy}=\frac{dN}{dx}\ ,\ \frac{dM}{dz}=\frac{dP}{dx}\ ,\ \frac{dN}{dz}=\frac{dP}{dy}\ :$$

his expletis, methodus integrandi erit ipsa eadem quam exhibuimus (151 : b^{IX}). Etsi non explentur, adhuc tamen adhiberi poterit illa methodus, modo inveniatur ejusmodi factor φ per quem multiplicata (o_8), sicque habita

$$\varphi Mdx+\varphi Ndy+\varphi Pdz=0\ldots(o_9)\ ,$$

inde prodeant

$$\frac{d(\varphi M)}{dy}=\frac{d(\varphi N)}{dx}\ ,\ \frac{d(\varphi M)}{dz}=\frac{d(\varphi P)}{dx}\ ,\ \frac{d(\varphi N)}{dz}=\frac{d(\varphi P)}{dy}\ldots(o_{10})\ :$$

suppeditabit namque formula (b^{IX}: 151) integrale aequationis (o_9), quod manifeste pertinebit etiam ad (o_8). Verum si difficile est invenire factorem quando duae sunt (161 1.°) variabiles, multo difficilius erit quando earum numerus $=3$; siquidem inveniri debet quantitas φ satisfaciens ternis

$$\left.\begin{aligned}
M\frac{d\varphi}{dy} - N\frac{d\varphi}{dx} + \varphi\left(\frac{dM}{dy} - \frac{dN}{dx}\right) = 0,\\
M\frac{d\varphi}{dz} - P\frac{d\varphi}{dx} + \varphi\left(\frac{dM}{dz} - \frac{dP}{dx}\right) = 0,\\
N\frac{d\varphi}{dz} - P\frac{d\varphi}{dy} + \varphi\left(\frac{dN}{dz} - \frac{dP}{dy}\right) = 0,
\end{aligned}\right\} \quad (o_{11})$$

provenientibus ex evolutione (o_{10}).

Prima (o_{11}) multiplicetur per P, secunda per $-N$, tertia per M, et in summam colligantur facta; exsurget

$$P\frac{dM}{dy} - M\frac{dP}{dy} - P\frac{dN}{dx} + N\frac{dP}{dx} - N\frac{dM}{dz} + M\frac{dN}{dz} = 0 \ldots (o_{12}):$$

hinc infertur, nisi valeat (o_{12}), nullum fore factorem qui (o_8) reddat integrabilem.

171. Si (o_{12}) prodit idemtica, integratio aequationis (o_8) unice pendebit ab integratione aequationum differentialium binas complectentium variabiles. Assertio sic potest ostendi: scribatur (o_8) in hunc modum

$$dz + \frac{N}{P}dy = -\frac{M}{P}dx \ldots (o_{13});$$

et denotante φ factorem illum per quem primum membrum (o_{13}) constituitur differentiale exactum, ponatur

$$\left(dz + \frac{N}{P}dy\right)\varphi = 0 \ldots (o_{14})$$

haud habita ratione secundi membri: erit

$$\int\left(dz + \frac{N}{P}dy\right)\varphi = \mu.$$

Facto compendii causa

$$\int(dz+\frac{N}{P}dy)\varphi=V\ldots(o_{15}),$$

differentietur aequatio

$$V=\mu\ldots(o_{16})$$

quoad x, y, z spectando μ ut functionem x; proveniet

$$\frac{dV}{dz}dz+\frac{dV}{dy}dy+\frac{dV}{dx}dx=\frac{d\mu}{dx}dx\ ,$$

seu

$$(dz+\frac{N}{P}dy)\varphi=-\frac{dV}{dx}dx+\frac{d\mu}{dx}dx\ ,$$

quam comparantes cum (o_{13}) multiplicata per φ assequimur

$$\frac{d\mu}{dx}=\frac{dV}{dx}-\varphi\frac{M}{P}\ldots(o_{17}).$$

Patet nunc illud: integratio aequationis (o_8) unice pendebit ab integratione aequationum differentialium binas complectentium variabiles, quotiescumque secundum membrum (o_{17}) traducetur (saltem postquam eliminata fuerit z ope aequationis (o_{16})) ad functionem quantitatum μ, x; ac proinde quotiescumque locus erit aequationi

$$\frac{d(\frac{dV}{dx}-\varphi\frac{M}{P})}{dy}=0\ ,$$

in qua ponenda est z variari ut functio quantitatis y. Aequationem istam evolventes obtinemus

$$\left.\begin{array}{l}\dfrac{d^2V}{dxdy}+\dfrac{d^2V}{dxdz}\cdot\dfrac{dz}{dy}-\varphi\left(\dfrac{d(\frac{M}{P})}{dy}+\dfrac{d(\frac{M}{P})}{dz}\cdot\dfrac{dz}{dy}\right)-\\ \qquad\dfrac{M}{P}\left(\dfrac{d\varphi}{dy}+\dfrac{d\varphi}{dz}\cdot\dfrac{dz}{dy}\right)=0\end{array}\right\}(o_{18}):$$

habemus autem ex (o_{15}) $\frac{dV}{dy} = \varphi\frac{N}{P}$, $\frac{dV}{dz} = \varphi$; ex (o_{13}) $\frac{dz}{dy} = -\frac{N}{P}$; primum insuper membrum (o_{18}) ponitur differentiale exactum, ideoque $\frac{d\varphi}{dy} = \frac{d(\varphi\frac{N}{P})}{dz}$:

itaque $\frac{d^2V}{dxdy} = \frac{d(\varphi\frac{N}{P})}{dx} = \frac{N}{P}\cdot\frac{d\varphi}{dx} + \varphi\frac{d(\frac{N}{P})}{dx}$,

$\frac{d^2V}{dxdz}\cdot\frac{dz}{dy} = -\frac{N}{P}\cdot\frac{d\varphi}{dx}$, $\frac{d\varphi}{dy} = \frac{N}{P}\cdot\frac{d\varphi}{dz} + \varphi\frac{d(\frac{N}{P})}{dz}$;

quibus valoribus substitutis in (o_{18}), pervenietur ad

$$\frac{d(\frac{N}{P})}{dx} - \frac{d(\frac{M}{P})}{dy} + \frac{N}{P}\,\frac{d(\frac{M}{P})}{dz} - \frac{M}{P}\,\frac{d(\frac{N}{P})}{dz} = 0,$$

et consequenter ad

$$P\frac{dM}{dy} - M\frac{dP}{dy} - P\frac{dN}{dx} + N\frac{dP}{dx} - N\frac{dM}{dz} + M\frac{dN}{dz} = 0 ;$$

quae cum sit eadem ac (o_{12}), jam patet veritas assertionis

172. Ex demonstratis (161 : 171) intelligimus, quotiescumque expletur (o_{12}), aequationem (o_8) fore integrabilem, ipsamque ab unica oriri primitiva (o_{16}); ideoque ad superficiem aliquam curvam pertinere: integrationis vero methodum eruimus ex ipso praecedentis (171) demonstrationis processu, spectando nimirum unam e variabilibus x, y, z tamquam constantem. Proponatur v. gr. integranda

$$(y^2+yz)dx+(xz+z^2)dy+(y^2-xy)dz=0:$$

erunt $M=y^2+yz$, $N=xz+z^2$; $P=y^2-xy$, qui valores reddunt $(o_{,2})$ idemticam. Habita nunc y pro constanti, poterit aequatio scribi in hunc modum

$$\frac{dx}{y^2-xy}+\frac{dz}{y^2+yz}=0;$$

et facta multiplicatione per y, sumptisque integralibus,

$$\frac{y+z}{y-x}=\mu\ldots(a):$$

haec differentiata quoad x, y, z dat

$$\frac{(y+z)dx-(x+z)dy+(y-x)dz}{(y-x)^2}=d\mu,$$

seu

$$\frac{(y^2+yz)dx-(xy+yz)dy+(y^2-xy)dz}{y(y-x)^2}=d\mu.$$

Jam hanc comparantes cum aequatione proposita assequimur

$$d\mu=-\frac{(xy+yz)dy+(xz+z^2)dy}{y(y-x)^2}=-\frac{(x+z)(y+z)dy}{y(y-x)^2};$$

e cujus secundo membro eliminanda est z ope aequationis (a): facta eliminatione, prodibit

$$d\mu=-\frac{\mu(\mu-1)}{y}dy,$$

unde

$$\frac{d\mu}{\mu(\mu-1)}=-\frac{dy}{y},\ \text{seu}\ \frac{d\mu}{\mu-1}-\frac{\mu}{d\mu}=-\frac{dy}{y};$$

ideoque $\frac{\mu-1}{\mu}=\frac{C}{y}$, $\mu=\frac{y}{y-C}$; et quaesitum in-

tegrale erit

$$\frac{y+z}{y-x}=\frac{y}{y-C}, \text{ quod potest ita scribi } \frac{xy+yz}{y+z}=C.$$

Si loco y habeatur z ut constans, expeditius determinabitur μ: nam proposita aequatio recipiet hanc formam

$$\frac{dx}{xz+z^2}+\frac{dy}{y^2+yz}=0,$$

seu

$$\frac{dx}{z(x+z)}+\frac{dy}{zy}-\frac{dy}{z(y+z)}=0.$$

Facta multiplicatione per z, sumptisque integralibus,

$$\frac{xy+zy}{y+z}=\mu:$$

haec differentietur, posita etiam z variabili; fiet

$$\frac{(y^2+yz)dx+(xy+z^2)dy+(y^2-xy)dz}{(y+z)^2}=d\mu;$$

quam conferentes aequationi propositae assequimur

$$d\mu=0, \text{ unde } \mu=C.$$

Quaesitum igitur integrale erit

$$\frac{xy+zy}{y+z}=C,$$

ut in hypothesi y constantis.

173. Conditione (o_{12}) haud expleta, etsi secundum membrum aequationis (o_{17}) minime redigitur ad functionem x, attamen aequationi (o_8) satisfiet per primitivam (o_{16}), modo una cum (o_{16}) ponatur valere ipsa quoque (o_{17}): facta nimirum $\mu=\chi(x)$, haud du-

hic explebitur (o_8) per systema binarum:

$$\left.\begin{array}{l} V = \chi(x), \\ \dfrac{dV}{dx} - \varphi\dfrac{M}{P} = \chi'(x), \end{array}\right\} (o_{19})$$

quaecumque caeteroquin sit forma functionis χ pro lubito sumendae. In ea igitur qua sumus hypothesi aequatio (o_8) non superficiei curvae, sed ob infinitas numero diversas formas functionis χ innumeris respondebit lineis duplicem habentibus curvedinem, et quadam praeditis communi proprietate expressa per ipsam (o_8).

Exempla.

I.° Sit

$$(z(z-a)+y(y-b))dx-(x-c)(zdz+ydy)=0 :$$

habemus

$M = z(z-a)+y(y-b)$, $N = -y(x-c)$, $P = -z(x-c)$;

quibus valoribus non expletur (o_{12}): facto autem $\varphi = z$, prodibit

$$V = \frac{z^2+y^2}{2} ;$$

explebitur itaque data aequatio per systema binarum

$$\frac{z^2+y^2}{2} = \chi(x), \quad \frac{z(z-a)+y(y-b)}{x-c} = \chi'(x).$$

II.° Detur

$$ydx+(z-x)dy+(x-z)dz=0 :$$

erunt $M=y$, $N=z-x$, $P=x-z$; quibus valoribus haud expletur (o_{12}); habemus autem $\varphi=1$, et $V=z-y$: quare per systema duarum

$$z-y=\chi(x)\,,\ -\frac{y}{x-z}=\chi'(x)$$

satisfiet aequationi datae.

174. Si M, N, P sunt homogeneae quoad x, y, z, poterit aequatio (o_8) hac ratione tractari: fiant $y=xr$, $z=xs$ ut substituantur hi valores in M, N, P; vertetur (o_8) in

$$M_1dx+N_1dy+P_1dz=0\,;$$

M_1, N_1, P_1 designant functiones variabilium r, s. Jamvero

$$dy=xdr+rdx\,,\ dz=xds+sdx:$$

igitur

$$(M_1+N_1r+P_1s)dx+(N_1dr+P_1ds)x=0\,;$$

et

$$\frac{dx}{x}+\frac{N_1dr+P_1ds}{M_1+N_1r+P_1s}=0\,;$$

cujus integrale eruetur ex dictis (151 : b^{XII}), modo expleatur aequatio

$$\frac{d(\frac{N_1}{M_1+N_1r+P_1s})}{ds}=\frac{d(\frac{P_1}{M_1+N_1r+P_1s})}{dr}\,.$$

Exemplum quisque potest desumere ab aequatione jam considerata (172).

175. In aequatione differentiali ponantur dx, dy, dz primum praetergredi gradum: ex ejus integrali, si quidem exsistit, eruetur certe per differentiationem aequatio, quae poterit ad formam (o_8) traduci. Hinc sequitur, nisi differentialis aequatio (postquam fuerit ordinata juxta potestates dz) resolvi queat in factores hujus formae

$$dz+pdx+qdy=0,$$

denotantibus p, q functiones variabilium x, y, z, sequitur inquam eam non fore integrabilem. Proponatur v. gr.

$$Pdx^2+Qdy^2+Rdz^2+2Sdxdy+2Tdxdz+2Vdydz=0:$$

factis compendii causa

$$T^2-PR=H,\ TV-RS=K,\ V^2-QR=L,$$

exsurget

$$dz=\frac{-Tdx-Vdy\pm\sqrt{(Hdx^2+2Kdxdy+Ldy^2)}}{R}.$$

Jam nisi termini sub signo radicali constituant perfectum quadratum, proposita aequatio neque in factores praedictam habentes formam poterit resolvi, neque proinde integrari; constituent autem perfectum quadratum, ubi fuerit

$$K=\sqrt{HL}\ \text{seu}\ K^2=HL.$$

176. Pertractatis aequationibus differentialibus, quae duas tresve complectuntur variabiles, aliquid nunc generatim subjungimus de iis, quae quatuor pluresve continent variabiles. Proponatur

$$Mdx+Ndy+Pdz+Qdu+Rdv+\ldots=0\ \ldots\ (o_{20}),$$

in qua primum membrum non est differentiale exactum (151), et M, N, P, ... exprimunt functiones variabilium x, y, z, u, v, ...; investigandae autem sint conditiones illae, quae debent expleri ut (o_{20}) derivari possit per differentiationem ex aliqua primitiva aequatione.

Si licet derivare (o_{20}) ab aliqua primitiva aequatione, licebit quoque considerare in (o_{20}) unam quamvis inter variabiles v. gr. z tamquam caeterarum functionem; eritque proinde

$$dz = \frac{dz}{dx}dx + \frac{dz}{dy}dy + \frac{dz}{du}du + \frac{dz}{dv}dv + \ldots;$$

et quoniam ipsa (o_{20}) praebet

$$dz = -\frac{M}{P}dx - \frac{N}{P}dy - \frac{Q}{P}du - \frac{R}{P}dv - \ldots,$$

iccirco

$$\left.\begin{array}{l} \frac{dz}{dx} = -\frac{M}{P}, \frac{dz}{dy} = -\frac{N}{P}, \\ \frac{dz}{du} = -\frac{Q}{P}, \frac{dz}{dv} = -\frac{R}{P}, \text{ et caet.} \end{array}\right\} (o_{21});$$

et consequenter

$$\left.\begin{array}{l} \frac{d(\frac{M}{P})}{dy} = \frac{d(\frac{N}{P})}{dx}, \frac{d(\frac{M}{P})}{du} = \frac{d(\frac{Q}{P})}{dx}, \\ \frac{d(\frac{M}{P})}{dv} = \frac{d(\frac{R}{P})}{dx}, \ldots \frac{d(\frac{N}{P})}{du} = \frac{d(\frac{Q}{P})}{dy}, \\ \frac{d(\frac{N}{P})}{dv} = \frac{d(\frac{R}{P})}{dy}, \ldots \frac{d(\frac{Q}{P})}{dv} = \frac{d(\frac{R}{P})}{du}, \ldots \end{array}\right\} (o_{22})$$

Jam evolutis (o_{22}), et loco $\frac{dz}{dx}$, $\frac{dz}{dy}$, $\frac{dz}{du}$, ... adhibitis valoribus ex (o_{21}), prodibunt conditiones explendae ut possit (o_{20}) derivari et caet. . . .

Hinc si variabiles sunt numero tres, unam dumtaxat habebimus conditionem

$$\frac{d(\frac{M}{P})}{dy} = \frac{d(\frac{N}{P})}{dx},$$

eamdem videlicet ac $(o_{1,2})$. Si variabiles sunt numero quatuor, ternas obtinebimus conditiones

$$\frac{d(\frac{M}{P})}{dy} = \frac{d(\frac{N}{P})}{dx}, \quad \frac{d(\frac{M}{P})}{du} = \frac{d(\frac{Q}{P})}{dx}, \quad \frac{d(\frac{N}{P})}{du} = \frac{d(\frac{Q}{P})}{dy};$$

idest

$$\left.\begin{array}{l} N\frac{dP}{dx} - P\frac{dN}{dx} + P\frac{dM}{dy} - M\frac{dP}{dy} + M\frac{dN}{dz} - \\ N\frac{dM}{dz} = 0, \; M\frac{dP}{du} - P\frac{dM}{du} + P\frac{dQ}{dx} - Q\frac{dP}{dx} + \\ Q\frac{dM}{dz} - M\frac{dQ}{dz} = 0, \; N\frac{dP}{du} - P\frac{dN}{du} + \\ P\frac{dQ}{dy} - Q\frac{dP}{dy} + Q\frac{dN}{dz} - N\frac{dQ}{dz} = 0. \end{array}\right\} (o_{2,3})$$

Generatim si variabilium numerus $= n$, erit conditionum numerus $= \frac{(n-1)(n-2)}{2}$.

177. Attendenti patebit methodum jam declaratam (172) aptari aequationibus differentialibus, quae quatuor pluresve complectuntur variabiles: exemplum praebeat aequatio

$$z(y+z)dx + z(u-x)dy + y(x-u)dz + y(y+z)du = 0.$$

Habemus $M = yz + z^2$, $N = zu - xz$, $P = xy - yu$, $Q = y^2 + yz$; qui valores explent conditiones $(o_{2,3})$: ponantur z, u constantes; aequatio fiet

$$\frac{dx}{u-x}+\frac{dy}{y+z}=0,\ \text{cujus integrale}\ \frac{y+z}{u-x}=\mu;$$

denotat μ functionem z, u. Differentiando quoad x, y, z, u habemus

$$\frac{(y+z)dx+(u-x)dy+(u-x)dz-(y+z)du}{(u-x)^2}=d\mu:$$

sed ex data aequatione profluit

$$(y+z)dx+(u-x)dy=\frac{y(u-x)dz-y(y+z)du}{z};$$

igitur

$$\frac{y+z}{u-x}dz-\left(\frac{y+z}{u-x}\right)^2du=zd\mu,\ \text{seu}\ \mu dz-\mu^2du=zd\mu.$$

Hinc

$$\frac{\mu dz-zd\mu}{\mu^2}=du;\ \text{sumptisque integralibus},$$

$$\frac{z}{\mu}=u+C,\ \text{et}\ \mu=\frac{z}{u+C}:$$

quare integrale aequationis datae erit

$$\frac{y+z}{u-x}=\frac{z}{u+C}.$$

178. Veniunt quandoque integrandae duae pluresve simul aequationes, quarum numerum unitate superat numerus variabilium x, y, z, ... Dentur

$$\left.\begin{array}{l}(a_1z+b_1y)dx+h_1dz+k_1dy=X_1dx,\\ (a_2z+b_2y)dx+h_2dz+k_2dy=X_2dx,\end{array}\right\}(o_{24})$$

ut inveniantur ejusmodi valores y, z expressi per x,

quibus eae expleantur ambae : $a_1, \ldots k_1, a_2, \ldots k_2$ denotant quantitates constantes, X_1 vero et X_2 functiones x. Si ab (o_{24}) eliminatur prius dy, dein dz, et quae resultant aequationes dividuntur per $h_1k_2-h_2k_1$, positis brevitatis gratia

$$\left.\begin{array}{l} \dfrac{a_1k_2-a_2k_1}{h_1k_2-h_2k_1}=A_1,\ \dfrac{b_1k_2-b_2k_1}{h_1k_2-h_2k_1}=B_1,\ \dfrac{X_1k_2-X_2k_1}{h_1k_2-h_2k_1}=X'_1, \\ \dfrac{a_2h_1-a_1h_2}{h_1k_2-h_2k_1}=A_2,\ \dfrac{b_2h_1-b_1h_2}{h_1k_2-h_2k_1}=B_2,\ \dfrac{X_2h_1-X_1h_2}{h_1k_2-h_2k_1}=X'_2, \end{array}\right\} (o_{25})$$

habebimus

$$\left.\begin{array}{l} (A_1z+B_1y)dx+dz=X'_1dx \\ (A_2z+B_2y)dx+dy=X'_2dx: \end{array}\right\} (o_{26})$$

istarum altera multiplicetur per indeterminatam ω, factumque inde proveniens addatur alteri ; erit

$$\left.\begin{array}{l} (B_1\omega+B_2)\left(\dfrac{A_1\omega+A_2}{B_1\omega+B_2}z+y\right)dx+\omega dx+dy= \\ \qquad (X'_1\omega+X'_2)dx. \end{array}\right\} (o_{27})$$

Jam si ad ω determinandam ponatur

$$\frac{A_1\omega+A_2}{B_1\omega+B_2}=\omega,$$

exsistet

$$\omega=\frac{A_1-B_2\pm\sqrt{[4A_2B_1+(A_1-B_2)^2]}}{2B_1} \ldots (o_{28});$$

quibus valoribus ω exhibitis per ω', ω'', iisque substitutis in (o_{27}), prodibunt

$$(B_1\omega'+B_2)(\omega'z+y)dx+d(\omega'z+y)=(X'_1\omega'+X'_2)dx,$$

$$(B_1\omega''+B_2)(\omega''z+y)dx+d(\omega''z+y)=(X'_1\omega''+X'_2)dx.$$

Hae praebent (156 : I.°)

$$\left.\begin{aligned}\omega' z+y&=e^{-(B_1\omega'+B_2)x}[C+\int\frac{X'_1\,\omega'+X'_2}{e^{-(B_1\omega'+B_2)x}}dx],\\ \omega'' z+y&=e^{-(B_1\omega''+B_2)x}[C_1+\int\frac{X'_1\,\omega''+X'_2}{e^{-(B_1\omega''+B_2)x}}dx];\end{aligned}\right\}(o_{29})$$

unde y et z.

179. Si $4A_2B_1+(A_1-B_2)^2=0$, ideoque $\omega'=\omega''$, binae (o_{29}) recident in unam: verum in ea qua sumus hypothesi exhibito compendii causa per $\chi(x)$ secundo membro alterutrius (o_{29}) ut sit

$$y=\chi(x)-\omega' z\ \ldots\ (o_{30}),$$

atque hoc valore y substituto in prima (o_{26}), exsurget

$$dz+(A_1-\omega')zdx=(X'_1-B_1\chi(x))dx,$$

cujus integratio (156 : I.°) suppeditat

$$z=e^{(\omega'-A_1)x}[C_1+\int\frac{(X'_1-B_1\chi(x))dx}{e^{(\omega'-A_1)x}}]\ \ldots\ (o_{31}).$$

Determinata z, assequemur y ex (o_{30}).

Exempla.

I.°

$$(44z+49y)dx+4dz+9dy=xdx,$$
$$(34z+38y)dx+3dz+7dy=e^x dx.$$

Erunt (o_{24})

$$a_1=44,\ b_1=49,\ h_1=4,\ k_1=9,$$

$$X_1=x,\ a_2=34,\ b_2=38,\ h_2=3,\ k_2=7,\ X_2=e^x;$$

et consequenter (o_{21})

$$A_1=2\,,\ B_1=1\,,\ X'_1=7x-9e^x\,,$$

$$A_2=4\,,\ B_2=5\,,\ X'_2=4e^x-3x\,;$$

ideoque (o_{22})

$$\omega'=1\,,\ \omega''=-4.$$

Binae igitur (o_{29}) evadunt (125)

$$z+y=\frac{21}{3}x-\frac{5}{7}e^x+Ce^{-6x}-\frac{1}{9}\,,$$

$$y-4z=20e^x-31x+C_1e^{-x}+31\,;$$

proinde

$$y=\frac{24}{7}e^x-\frac{17}{3}x+4Ce^{-6x}+C_1e^{-x}+\frac{55}{9}\,,$$

$$z=\frac{19}{3}x-\frac{29}{7}e^x+Ce^{-6x}-C_1e^{-x}-\frac{56}{9}\,.$$

II.°

$$(11z+31y)dx+4dz+9dy=e^xdx\,,$$

$$(8z+24y)dx+3dz+7dy=e^{2x}dx.$$

Quoniam (o_{24})

$$a_1=11\,,\ b_1=31\,,\ h_1=4\,,\ k_1=9\,,\ X_1=e^x\,,$$

$$a_2=8\,,\ b_2=24\,,\ h_2=3\,,\ k_2=7\,,\ X_2=e^{2x}\,,$$

iccirco (o_{25})

$$A_1=5\,,\ B_1=1\,,\ X'_1=7e^x-9e^{2x}\,,$$

$$A_2=-1\,,\ B_2=3\,,\ X'_2=4e^{2x}-3e^x\,;$$

unde (o_{28})

$$\omega'=\omega''=1.$$

Est autem (o_{29})

$$\chi(x)=\frac{4}{5}e^{x}-\frac{5}{6}e^{2x}+Ce^{-4x};$$

igitur (o_{31} : o_{30})

$$z=\frac{31}{25}e^{x}-\frac{49}{36}e^{2x}-Cxe^{-4x}+C_1e^{-4x},$$

$$y=\frac{19}{36}e^{2x}-\frac{11}{25}e^{x}+\frac{C(x+1)-C_1}{e^{4x}}.$$

180. Si $b_1k_2-b_2k_1=0$, erit $B_1=0$, et ω', ω'' infinitae; at prima (o_{26}) in ea qua sumus hypothesi praebet

$$A_1zdx+dz=X'_1dx,$$

ex qua (156 : I.°)

$$z=e^{-A_1x}(C+\int e^{A_1x}X'_1dx)\ldots(o_{32}):$$

secundum membrum (o_{32}) designa per $\chi(x)$, illudque substitue loco z in secunda (o_{26}); habebis

$$B_2ydx+dy=(X'_2-A_2\chi(x))dx,$$

unde (156 : I.°)

$$y=e^{-B_2x}(C_1+\int e^{B_2x}(X'_2-A_2\chi(x))dx)\ldots(o_{33}).$$

181. Sint nunc integrandae tres simul aequationes

$$\left.\begin{array}{l}(a_1y+b_1z+h_1t)dx+i_1dy+k_1dz+m_1dt=X_1dx,\\(a_2y+b_2z+h_2t)dx+i_2dy+k_2dz+m_2dt=X_2dx,\\(a_3y+b_3z+h_3t)dx+i_3dy+k_3dz+m_3dt=X_3dx.\end{array}\right\}(o_{34})$$

Eliminatis ab (o_{34}) 1.° dy et dz, 2.° dz et dt,

3.° dy et dt, aequationes inde prodeuntes erunt ejusmodi formae

$$\left.\begin{array}{l}(A_1 y+B_1 z+H_1 t)dx+dt=X'dx\,,\\ (A_2 y+B_2 z+H_2 t)dx+dy=X''dx\,,\\ (A_3 y+B_3 z+H_3 t)dx+dz=X'''dx.\end{array}\right\}(o_{35})$$

Multiplica secundam (o_{35}) per indeterminatam α, tertiam per ω similiter indeterminatam, et producta adde primae; exsistet

$$\left.\begin{array}{c}(H_1+H_2\alpha+H_3\omega)(\dfrac{A_1+A_2\alpha+A_3\omega}{H_1+H_2\alpha-H_3\omega}y+\\ \dfrac{B_1+B_2\alpha+B_3\omega}{H_1+H_2\alpha+H_3\omega}z+t)dx+\alpha dy+\omega dz+dt=\\ (X'+X''\alpha+X'''\omega)dx.\end{array}\right\}(o_{36}).$$

Ad α et ω determinandas ponantur

$$\frac{A_1+A_2\alpha+A_3\omega}{H_1+H_2\alpha+H_3\omega}=\alpha\,,\ \frac{B_1+B_2\alpha+B_3\omega}{H_1+H_2\alpha+H_3\omega}=\omega\,;$$

eruetur ex prima

$$\omega=\frac{(H_1-A_2)\alpha+H_2\alpha^2-A_1}{A_3-H_3\alpha}\ \ldots\ (o_{37})\,;$$

et substituto valore ω in secunda, proveniet aequatio tertii gradus in ordine ad α, unde terni valores α', α'', α'''; ideoque ex (o_{37}) terni pariter valores ω', ω'', ω'''. Habemus itaque ex (o_{36})

$$(H_1+H_2\alpha'+H_3\omega')(\alpha' y+\omega' z+t)dx+$$

$$d(\alpha' y+\omega' z+t)=(X'+X''\alpha'+X'''\omega')dx\,,$$

$$(H_1+H_2\alpha''+H_3\omega'')(\alpha'' y+\omega'' z+t)dx+$$

$$d(\alpha''+y\omega'' z+t)=(X'+X''\alpha''+X'''\omega'')dx\,,$$

$$(\mathrm{H}_1+\mathrm{H}_2\alpha'''+\mathrm{H}_3\omega''')(\alpha'''y+\omega'''z+t)dx+$$

$$d(\alpha'''y+\omega'''z+t)=(\mathrm{X}'+\mathrm{X}''\alpha'''+\mathrm{X}'''\omega''')dx,$$

quae suppeditabunt (156 : I.°)

$$\alpha'y+\omega'z+t=\mathrm{F}_1(x)\ ,\ \alpha''y+\omega''z+t=\mathrm{F}_2(x)\ ,$$

$$\alpha'''y+\omega'''z+t=\mathrm{F}_3(x)\ ;$$

unde y, z, t expressae per x.

DE INTEGRATIONE QUARUMDAM AEQUATIONUM DIFFERENTIALIUM QUAE PRIMUM EXCEDUNT ORDINEM.

182. Proponatur 1.°

$$\frac{d^2y}{dx^2}=f(y)\ldots(g):$$

facto $dx=vdy$, prodibit $d^2x=0=vd^2y+dvdy$, unde $d^2y=-\frac{dvdy}{v}$. Substitutis in (g) valoribus dx, d^2y, proveniet

$$-\frac{dv}{v^3}=f(y)dy\text{, ideoque }\frac{1}{2v^2}=\int f(y)dy+\mathrm{C}:$$

erit igitur

$$v=\pm\frac{1}{\sqrt{(2\int f(y)dy+\mathrm{C}_1)}},$$

et consequenter

$$dx=\pm\frac{dy}{\sqrt{(2\int f(y)dy+\mathrm{C}_1)}}.$$

2.°

$$\frac{d^2y}{dx^2}=f\left(x\,,\,\frac{dy}{dx}\right)\ldots(g').$$

facto $\frac{dy}{dx}=v$, unde $\frac{d^2y}{dx^2}=\frac{dv}{dx}$, adhibitisque substitutionibus in (g'), profluet $dv=f(x\,,\,v)dx$, cujus integrale suppeditabit $v=F(x\,,\,C)$. Hinc

$$dy=F(x\,,\,C)dx.$$

3.°

$$\frac{d^2y}{dx^2}=f(y\,,\,\frac{dy}{dx})\ldots(g'').$$

Positio (1.°) $dx=vdy$ mutat (g'') in

$$-\frac{dvdy}{v}=f(x\,,\,\frac{1}{v})dx^2:$$

est autem $dx^2=v^2dy^2$; hinc

$$-\frac{dv}{v^3}=f(y\,,\,\frac{1}{v})dy\,,$$

cujus integrale praebebit

$$v=F(y\,,\,C)\,;\ \text{proinde}\ dx=F(y\,,\,C)dy.$$

4.°

$$\frac{d^2y}{dx^2}=f(x\,,y\,,\frac{dy}{dx})\ldots(g'''),$$

quae ponitur homogenea quoad $x\,,y\,,dx\,,dy\,,d^2y$: factis

$$y=ux\,,\ \frac{dy}{dx}=v\,,\ \frac{d^2y}{dx^2}=\frac{z}{x}\,,$$

assequemur aequationem $V=0$ inter u, v et z. Jam vero

$$dy=vdx=udx+xdu\,,\ \frac{d^2y}{dx^2}=\frac{dv}{dx}=\frac{z}{x}\,,$$

ideoque

$$\left.\begin{aligned} \frac{dx}{x} &= \frac{du}{v-u}, \\ \frac{dv}{dz} &= \frac{du}{v-u} : \end{aligned}\right\} (k)$$

itaque in secunda (k) substituto valore z ex $V=0$, proveniet aequatio primi ordinis inter u et v, cujus integrale praebebit v expressam per u; et adhibita substitutione in prima (k), eruetur inde relatio inter x et u, ex qua eficietur u ope $y=ux$.

Exempla.

I.° $a^2d^2y+ydx^2=0$ seu $\frac{d^2y}{dx^2}=-\frac{y}{a^2}$: erit (1.°)

$f(y)=-\frac{y}{a^2}$, $\int f(y)dy = -\frac{y^2}{2a^2}$; unde

$$dx=\pm\frac{dy}{\sqrt{(C_1-\frac{y^2}{a^2})}},\ x=\pm\int\frac{dy}{\sqrt{(C_1-\frac{y^2}{a^2})}}+$$

$$C_2=\pm\frac{1}{C_1^{\frac{1}{2}}}\int\frac{dy}{\sqrt{(1-\frac{y^2}{C_1a^2})}}+C_2=$$

$$\pm\frac{1}{C_1^{\frac{1}{2}}}arc\left(sin=\frac{y}{aC_1^{\frac{1}{2}}}\right)+C_2.$$

II.°

$$\frac{d^2y}{dx^2}=\frac{(\frac{dy^2}{dx^2}+1)^{\frac{3}{2}}}{x^2}:$$

erit (2.°)

$$d\vartheta = \frac{(\nu^2+1)^{\frac{3}{2}}}{x^2}dx, \text{ ideoque } \frac{d\vartheta}{(\nu^2+1)^{\frac{3}{2}}} = \frac{dx}{x^2};$$

sumptisque integralibus (139),

$$C - \frac{1}{x} = \frac{\nu}{\sqrt{(\nu^2+1)}}, \quad \nu = \frac{Cx-1}{\sqrt{[x^2-(Cx-1)^2]}};$$

et consequenter

$$dy = \frac{(Cx-1)dx}{\sqrt{[x^2-(Cx-1)^2]}}.$$

III.°

$$\frac{d^2y}{dx^2} = \frac{1}{ab}\sqrt{(y^2 + a^2\frac{dy^2}{dx^2})}:$$

erit (3.°)

$$-\frac{d\vartheta}{\nu^3} = \frac{1}{ab}dy\sqrt{(y^2 + \frac{a^2}{\nu^2})};$$

et facto $\frac{1}{\nu} = \frac{y}{z}$,

$$abzdy - abydz = z^2dy\sqrt{(z^2 + a^2)},$$

$$\text{seu } \frac{dy}{y} = \frac{abdz}{z^2\sqrt{(z^2+a^2)}-abz}.$$

IV.°

$$\frac{d^2y}{dx^2} = \frac{1}{x^2}\sqrt{(ax^2\frac{dy^2}{dx^2} + by^2)}:$$

erit (4.°)

$$V = z - \sqrt{(a^2v^2 + bu^2)} = 0 ,$$

ideoque secunda (k) fiet

$$\frac{dv}{\sqrt{(a^2v^2+bu^2)}} = \frac{du}{v-u}, \text{ seu } (v-u)dv = du\sqrt{(a^2v^2+bu^2)},$$

et consequenter, assumpta (154 : 2.°) $v=\omega u$,

$$\frac{du}{u} = -\frac{(\omega-1)d\omega}{\omega(\omega-1)-\sqrt{(a^2\omega^2+b)}} \ldots (k').$$

Insuper prima (k) vertitur in

$$\frac{dx}{x} = \frac{du}{u(\omega-1)}; \text{ proinde } \frac{dx}{x} = -\frac{d\omega}{\omega(\omega-1)-\sqrt{(a^2\omega^2+b)}} \ldots (k''):$$

binae igitur (k'), (k'') suppeditabunt binas aequationes, alteram inter u et ω, seu inter $\frac{y}{x}$ et ω, alteram inter x et ω, e quibus demum ejicietur ω.

183. Etsi (g''') non est talis, qualem posuimus, attamen factis

$$y = ux^n, \; \frac{dy}{dx} = vx^{n-1}, \; \frac{d^2y}{dx^2} = zx^{n-2},$$

ubi liceat ita determinare n ut dispareat x, devenietur haud difficulter ad aequationem primi ordinis. Sit v. gr.

$$\frac{d^2y}{dx^2} = \frac{x^3+2xy}{x^4} \cdot \frac{dy}{dx} - \frac{4y^2}{x^4},$$

quae traducitur ad

$$(z-v)x^{n+2} + (4u^2-2uv)x^{2n} = 0.$$

Facto $n+2=2n$, ideoque $n=2$, disparebit x, eritque

$$z - v + 4u^2 - 2uv = 0 \ldots (h)$$

Jam vero ob $y=ux^2$ et $\frac{dy}{dx}=vx$ habemus

$$dy=2uxdx+x^2du\,,\ \frac{dy}{dx}=vx=2ux+x^2\frac{du}{dx}\,,$$

$$vdx=2udx+xdu\,;$$

proinde

$$vdx=2udx+xdu\,,\ \frac{dx}{x}=\frac{du}{v-2u}\ \ldots (h')\,;$$

insuper ob $\frac{d^2y}{dx^2}=\frac{1}{dx}d(vx)=z$ habemus

$$vdx+xdv=zdx\,,\ \frac{dx}{x}=\frac{dv}{z-v}\,:$$

igitur

$$\frac{du}{v-2u}=\frac{dv}{z-v}\,;$$

et substituto valore z ex (h),

$$dv=2udu\,,\ \text{ideoque}\ v=u^2+C.$$

Hinc (h')

$$\frac{dx}{x}=\frac{du}{u^2-2u+C}\,.$$

184. Si (g''') est homogenea quoad solas y, dx, dy, d^2y, positis

$$\frac{dy}{dx}=vy\,,\ \frac{d^2y}{dx^2}=zy\,,$$

assequemur aequationem inter x, v, z, ex qua $z=F(x, v)$. Sunt autem

$\frac{dy}{y}=vdx$, $\frac{1}{dx}d(vy)=zy$, ideoque $\frac{dy}{y}=\frac{zdx-dv}{v}$; igitur $F(x, v)dx-dv=v^2dx$, cujus integrale suppeditabit $v=f(x, C)$, et consequenter

$$\frac{dy}{y}=f(x, C)dx.$$

Detur v. gr.

$$\frac{d^2y}{dx^2}=\frac{1}{x}\cdot\frac{dy}{dx}+\frac{1}{y}\cdot\frac{dy^2}{dx^2}+\frac{b}{y\sqrt{(a^2-x^2)}}\cdot\frac{dy^2}{dx^2};$$

erit

$$z=\frac{v}{x}+v^2+\frac{bv^2}{\sqrt{(a^2-x^2)}};$$

proinde

$$\frac{vdx}{x}+v^2dx+\frac{bv^2dx}{\sqrt{(a^2-x^2)}}-dv=v^2dx,$$

quae traducitur ad

$$\frac{vdx-xdv}{v^2}+\frac{bxdx}{\sqrt{(a^2-x^2)}}=0 \text{ seu } d\left(\frac{x}{v}\right)-b.d\sqrt{(a^2-x^2)}=0.$$

Hinc

$$\frac{x}{v}-b\sqrt{(a^2-x^2)}=C,\ v=\frac{x}{C+b\sqrt{(a^2-x^2)}};$$

et consequenter

$$\frac{dy}{y}=\frac{xdx}{C+b\sqrt{(a^2-x^2)}}.$$

185. Bonum erit hic subjungere illud : integrationi expedit apte sumere differentiale habendum ut constans : rem declarabunt

Exempla.

I.° Sit aequatio

$$(a+x)\frac{d^2y}{dx^2}=\frac{dy}{dx}-\frac{dy^3}{dx^3}:$$

a suppositione dx constantis transeamus ad hypothesim nullius constantis; aequatio immutabitur (15) in

$$(a+x)\frac{dxd^2y-dyd^2x}{dx^3}=\frac{dy}{dx}-\frac{dy^3}{dx^3}:$$

hanc modo in eam convertamus, quae supponit dy constantem; erit

$$-(a+x)\frac{dyd^2x}{dx^3}=\frac{dy}{dx}-\frac{dy^3}{dx^3},$$

seu

$$xd^2x+dx^2-dy^2=-ad^2x,$$

quae integrata in hypothesi dy constantis dat

$$xdx-ydy=Cdy-adx;$$

et iterata integratione,

$$\frac{x^2-y^2}{2}=Cy-ax+C_1.$$

II.° Detur aequatio

$$xy(dxd^2y-dyd^2x)=ydydx^2-y^2df(y)dy^2-xdxdy^2,$$

in qua neque dx ponitur constans, neque dy: aequationem ita scribe

$$(xyd^2x+ydx^2-xdxdy)dy=y(xdxd^2y+ydf(y)dy^2),$$

ut traducatur ad

$$^2d(\frac{xdx}{y})dy=y(xdxd^2y+ydf(y)dy^2.$$

Pone $\frac{xdx}{y}$ constantem ; habebis

$$xdxd^2y+ydf(y)dy^2=0\text{ , seu }df(y)=-\frac{xdx}{y}\cdot\frac{d^2y}{dy^2}\ ;$$

atque integrando in hypothesi $\frac{xdx}{y}$ constantis,

$$f(y)=\frac{xdx}{y}\cdot\frac{1}{dy}+C\text{ , seu }(f(y)-C)ydy=xdx.$$

III.° Sit demum

$$x^2dyd^2x+x^2yd^3x-xd^3y+dxd^2y=x^2dxd^2x+x^2dyd^2y,$$

quam ita scribo

$$x^2(dyd^2x+yd^3x)-x^2d(\frac{d^2y}{x})=x^2(dxd^2x+dyd^2y):$$

accipio $\frac{d^2y}{x}$ tamquam constantem ; prodit

$$dyd^2x+yd^3x=dxd^2x+dyd^2y\ ,$$

cujus integrale

$$yd^2x+C\frac{d^2y}{x}=\frac{dx^2+dy^2}{2}\ .$$

186. Sit 1.°

$$\frac{d^ny}{dx^n}=f(\frac{d^{n-1}y}{dx^{n-1}})\ldots(g^{IV})$$

Facto $\frac{d^{n-1}y}{dx^{n-1}}=v$, vertetur (g^{IV}) in $\frac{dv}{f(v)}=dx$; ex qua integrata eruetur $v=F(x\text{ , }C)$. Hinc

$$d^{n-1}y=F(x\text{ , }C)dx^{n-1}\ ,$$

cujus ulterior integratio habetur ex dictis (148).

2.°

$$\frac{d^n y}{dx^n} = f\left(\frac{d^{n-2} y}{dx^{n-2}}\right) \dots (g^{\text{v}}):$$

facto $\nu = \frac{d^{n-2}y}{dx^{n-2}}$, unde $\frac{d\nu}{dx} = \frac{d^{n-1}y}{dx^{n-1}}$, $\frac{d^2\nu}{dx^2} = \frac{d^n y}{dx^n}$;

adhibitisque substitutionibus in (g^{v}), proveniet

$$\frac{d^2\nu}{dx^2} = f(\nu),$$

quae similiter integratur ac (g : 182 1.°). Sit itaque $\nu = F(x, C, C_1)$; erit

$$d^{n-2}y = F(x, C, C_1)dx^{n-2}.$$

Exempla.

I.°

$$\frac{d^4 y}{dx^4} = a\sqrt{\frac{d^3 y}{dx^3}}:$$

erit (1.°)

$$\frac{d\nu}{\sqrt{\nu}} = a dx, \; 2\sqrt{\nu} = ax + C, \; \nu = \frac{(ax+C)^2}{4}; \text{ proinde}$$

$$d^3 y = \frac{(ax+C)^2}{4} dx^3.$$

Jam si in (i^{VI} : 148) assumitur $x_0 = 0$, proveniet

$$y = \frac{x^3}{24}\left(\frac{a^2x^2}{10} + \frac{aCx}{2} + C^2\right) + \frac{C_1 x^2}{2} + C_2 x + C_3.$$

II.°

$$\frac{d^4y}{dx^4}=-\frac{a}{(\frac{d^2y}{dx^2})^3}:$$

erit (2.°)

$\frac{d^2v}{dx^2}=-\frac{a}{v^3}$; et facto $dx=zdv$, exsurget (182 : 1.°)

$$\frac{dz}{z^3}=\frac{adv}{v^3}.$$

Hinc

$$\frac{1}{z^2}=\frac{a}{v^2}+C\,,\; z=\frac{v}{\sqrt{(a+Cv^2)}},\; dx=\frac{vdv}{\sqrt{(a+Cv^2)}},$$

$x=\frac{\sqrt{(a+Cv^2)}}{C}+C_1$, $v=\sqrt{(\frac{(Cx-C_1)^2-a}{C})}$; demum (2.°)

$$d^2y=dx^2\sqrt{(\frac{(Cx-C_1)^2-a}{C})}.$$

187. Sit nunc

$$\frac{d^ny}{dx^n}+a_1\frac{d^{n-1}y}{dx^{n-1}}+a_2\frac{d^{n-2}y}{dx^{n-2}}+\ldots+a_{n-1}\frac{dy}{dx}+a_ny=f(x)\ldots(g^{VI});$$

a_1, a_2, $\ldots a_n$ denotant quantitates constantes. Facta

$$F(r)=r^n+a_1r^{n-1}+a_2r^{n-2}+\ldots+a_{n-1}r+a_n=(r-i_1)(r-i_2)\ldots(r-i_n),$$

et designata per $\chi(r\,,\,x)$ ejusmodi functione arbitraria, quae haud evadat infinita quum pro r adhibentur i_1, i_2, $\ldots$ i_n vel quivis alius finitus valor, sumatur (66)

$$y=\mathcal{E}\frac{\chi(r\,,\,x)\,.\,e^{rx}}{((F(r)))}\ldots(b).$$

Accepto insuper $m<n-1$, ponatur

$$\mathcal{E}e^{rx}\frac{d\chi(r\,,\ x)}{dx}\,\frac{r^m}{((\mathrm{F}(r)))}=0\ldots(b');$$

erunt (70)

$$\frac{dy}{dx}=\mathcal{E}e^{rx}\chi(r,x)\frac{r}{((\mathrm{F}(r)))}+\mathcal{E}e^{rx}\frac{d\chi(r,x)}{dx}\,\frac{1}{((\mathrm{F}(r)))}=$$

$$\mathcal{E}e^{rx}\chi(r,x)\frac{r}{((\mathrm{F}(r)))}\,,\ \frac{d^2y}{dx^2}=\mathcal{E}e^{rx}\chi(r\,,\ x)\,\frac{r^2}{((\mathrm{F}(r)))}+$$

$$\mathcal{E}e^{rx}\frac{d\chi(r\,,\ x)}{dx}\,\frac{r}{((\mathrm{F}(r)))}=\mathcal{E}e^{rx}\chi(r\,,\ x)\,\frac{r^2}{((\mathrm{F}(r)))}\,,$$

et caet. . . . ,

$$\frac{d^{n-1}y}{dx^{n-1}}=\mathcal{E}e^{rx}\chi(r,x)\frac{r^{n-1}}{((\mathrm{F}(r)))}+\mathcal{E}e^{rx}\frac{d\chi(r,\ x)}{dx}\,\frac{r^{n-2}}{((\mathrm{F}(r)))}=$$

$$\mathcal{E}e^{rx}\chi(r,x)\frac{r^{n-1}}{((\mathrm{F}(r)))}\,,\ \frac{d^ny}{dx^n}=\mathcal{E}e^{rx}\chi(r\,,x)\ \frac{r^n}{((\mathrm{F}(r)))}+$$

$$\mathcal{E}e^{rx}\frac{d\chi(r\,,x)}{dx}\,\frac{r^{n-1}}{((\mathrm{F}(r)))}\ ;$$

factisque substitutionibus in (g^{VI}),

$$\mathcal{E}\frac{e^{rx}\chi(r\,,\ x)\,\mathrm{F}(r)}{((\mathrm{F}(r)))}+\mathcal{E}e^{rx}\frac{d\chi(r\,,\ x)}{dx}\,\frac{r^{n-1}}{((\mathrm{F}(r)))}=f(x):$$

et quoniam binarum e^{rx}, $\chi(r\,,\ x)$ neutra fit infinita, quicumque ex valoribus $i_1\,,\ i_2\,,\ldots i_n$ substituitur loco r, iccirco (67. 2.°)

$$\mathcal{E}\frac{e^{rx}\chi(r\,,\ x)\,\mathrm{F}(r)}{((\mathrm{F}(r)))}=0\,,$$

et consequenter

$$\Sigma e^{rx}\frac{d\chi(r,x)}{dx}\,\frac{r^{n-1}}{((\mathrm{F}(r)))}=f(x)\ldots(b''):$$

satisfiet igitur per (b) aequationi (g^{vi}), modo $\chi(r, x)$ determinetur ita, ut (b'), (b'') simul expleantur. Atqui (69. 5.°)

$$\Sigma\frac{r^m}{((\mathrm{F}(r)))}=0\,,\ \Sigma\frac{r^{n-1}}{((\mathrm{F}(r)))}=1\,;$$

explebuntur itaque (b') et (b'') sumendo

$$e^{rx}\frac{d\chi(r,x)}{dx}=f(x),\text{ ex qua (151) }\chi(r,x)=\int_{x_0}^{x}e^{-rz}f(z)dz+\varphi(r):$$

hinc

$$y=\Sigma\frac{e^{rx}\varphi(r)}{((\mathrm{F}(r)))}+\Sigma\frac{\int_{x_0}^{x}e^{r(x-z)}f(z)dz}{((\mathrm{F}(r)))}\ldots(g_1).$$

Ad haec: cum $\varphi(r)$ sit arbitraria, continebuntur in (g_1) n arbitrariae $\varphi(i_1)$, $\varphi(i_2)$, ... $\varphi(i_n)$: neque arbitrariarum functionum numerus minuitur ob $i_1=i_2=$ et caet. ...; nam si

$$\mathrm{F}(r)=(r-i_1)^m\ldots(r-i_{n-1})(r-i_n),$$

deerunt sane $\varphi(i_2)$; $\varphi(i_3)$, ... $\varphi(i_m)$, verum quia (67. 4.°)

$$\Sigma\frac{e^{rx}\varphi(r)}{(([r-i_1]^m))\ldots(r-i_n)}=\frac{1}{1.2.3\ldots(m-1)}\,\frac{d^{m-1}\left[\frac{e^{(i_1+\theta)x}\varphi(i_1+\theta)}{(i_1+\theta-i_{m+1})\ldots(i_1+\theta-i_n)}\right]}{d\theta^{m-1}},$$

ideo illarum loco sese offerent $\varphi'(i_1)$, $\varphi''(i_1)$, ... $\varphi^{(m-1)}(i_1)$. Quibus positis, pronum est concludere fo-

re (g_1) integrale completum aequationis (g^{VI}). Si ponitur $f(x)=0$, ut habeamus

$$\frac{d^n y}{dx^n}+a_1\frac{d^{n-1}y}{dx^{n-1}}+a_2\frac{d^{n-2}y}{dx^{n-2}}+\ldots+a_{n-1}\frac{dy}{dx}+a_n y=0 \ldots (g^{VII}),$$

vertetur (g_1) in

$$y=\mathcal{E}\frac{e^{rx}\varphi(r)}{((F(r)))}\ldots(g_2).$$

Exempla.

I.°

$$\frac{d^3y}{dx^3}-14\frac{d^2y}{dx^2}+64\frac{dy}{dx}-96y=0:$$

erunt $n=3$, $a_1=-14$, $a_2=64$, $a_3=-96$, $f(x)=0$,

$F(r)=r^3-14r^2+64r-96=(r-6)(r-4)^2$; ideo

$$y=\mathcal{E}\frac{e^{rx}\varphi(r)}{((r-6))(r-4)^2}+\mathcal{E}\frac{e^{rx}\varphi(r)}{(r-6)(((r-4)^2))}.$$

Sed

$$\mathcal{E}\frac{e^{rx}\varphi(r)}{((r-6))(r-4)^2}=\frac{e^{6x}\varphi(6)}{4},\ \mathcal{E}\frac{e^{rx}\varphi(r)}{(r-6)(((r-4)^2))}=$$

$$\frac{d\left[\dfrac{\theta^2 e^{(4+\theta)x}\varphi(4+\theta)}{\theta^2(-2)}\right]}{d\theta}=-\tfrac{1}{2}\left(e^{4x}x\varphi(4)+e^{4x}\varphi'(4)\right);$$

factis igitur $\frac{1}{4}\varphi(6)=C_1$, $-\frac{1}{2}\varphi(4)=C_2$, $-\frac{1}{2}\varphi'(4)=C_3$,

proveniet $y=e^{4x}(C_1e^{2x}+C_2x+C_3)$.

II.°

$$\frac{d^2y}{dx^2} - i^2y = f(x):$$

erunt $n=2$; $a_1=0$, $a_2=-i^2$, $F(r)=r^2-i^2=(r+i)(r-i)$; proinde

$$y = \mathcal{E}\frac{e^{rx}\varphi(r)}{(((r+i)(r-i)))} + \mathcal{E}\frac{\int_{x_0}^{x} e^{r(x-z)}f(z)dz}{(((r+i)(r-i)))}.$$

Sed

$$\mathcal{E}\frac{e^{rx}\varphi(r)}{(((r+i)(r-i)))} = -\frac{e^{-ix}\varphi(-i)}{2i} + \frac{e^{ix}\varphi(i)}{2i};$$

praeterea (136. 2.°)

$$\mathcal{E}\frac{\int_{x_0}^{x} e^{r(x-z)}f(z)dz}{(((r+i)(r-i)))} = \int_{x_0}^{x}\frac{e^{-i(x-z)}f(z)dz}{-2i} +$$

$$\int_{x_0}^{x}\frac{e^{i(x-z)}f(z)dz}{2i}:$$

igitur

$$y = \frac{C_1e^{ix} - C_2e^{-ix}}{2i} + \frac{1}{2i}\int_{x_0}^{x}[e^{i(x-z)} - e^{-i(x-z)}]f(z)dz.$$

Si $-i^2=a$, ut veniat integranda

$$\frac{d^2y}{dx^2} + ay = f(x),$$

erit $i=\sqrt{a}\sqrt{-1}$: quare

$$y=\frac{C_1 e^{x\sqrt{a}\sqrt{-1}}-C_2 e^{-x\sqrt{a}\sqrt{-1}}}{2\sqrt{a}\sqrt{-1}}+$$

$$\frac{1}{2\sqrt{a}\sqrt{-1}}\int_{x_0}^{x}[e^{(x-z)\sqrt{a}\sqrt{-1}}-e^{-(x-z)\sqrt{a}\sqrt{-1}}]f(z)dz;$$

quae, adhibito $C_1+C_2\sqrt{-1}$ pro C_1 et $C_1-C_2\sqrt{-1}$ pro C_2, mutabitur (162. 1.° ex p. 2.^{a}) in

$$y=\frac{C_1 \sin x\sqrt{a}+C_2 \cos x\sqrt{a}}{\sqrt{a}}+\frac{1}{\sqrt{a}}\int_{x_0}^{x}\sin(x-z)\sqrt{a}.f(z)dz.$$

188. Sit etiam

$$\left.\begin{array}{l}\dfrac{d^n y}{dx^n}+\dfrac{a_1}{hx+k}\dfrac{d^{n-1}y}{dx^{n-1}}+\dfrac{a_2}{(hx+k)^2}\dfrac{d^{n-2}y}{dx^{n-2}}+\ldots \\ \quad +\dfrac{a_{n-1}}{(hx+k)^{n-1}}\dfrac{dy}{dx}+\dfrac{a_n y}{(hx+k)^n}=f(x)\end{array}\right\}(g^{\text{VIII}})$$

in qua non solum $a_1, a_2, \ldots a_n$ sed et h, k designant quantitates constantes: fiat

$$F(r)=h^n r(r-1)\ldots(r-n+1)+$$
$$a_1 h^{n-1} r(r-1)\ldots(r-n+2)+\ldots+a_{n-1}hr+a_n;$$

exprimat $\chi(r, x)$ ejusmodi functionem arbitrariam, quae minime evadat infinita quum pro r adhibentur valores aequationi $F(r)=0$ satisfacientes, vel quivis alius finitus valor; et assumpto $m<n-1$, ponamus

$$y=\mathcal{E}\frac{(hx+k)^r\chi(r, x)}{((F(r)))}\ldots(b^{\text{IV}}),$$

$$\mathcal{E}(hx+k)^{r-m}\frac{d\chi(r, x)}{dx}\frac{r(r-1)\ldots(r-m+1)}{((F(r)))}=0\ldots(b^{\text{V}}):$$

erunt

$$\frac{dy}{dx}=\Sigma(hx+k)^{r-1}\chi(r\,,x)\frac{hr}{((\mathrm{F}(r)))}\,,\ \frac{d^2y}{dx^2}=$$

$$\Sigma(hx+k)^{r-2}\chi(r\,,x)\,\frac{h^2r(r-1)}{((\mathrm{F}(r)))}\,,\ldots\frac{d^{n-2}y}{dx^{n-2}}=$$

$$\Sigma(hx+k)^{r-n+2}\chi(r,x)\frac{h^{n-2}r(r-1)\ldots(r-n+3)}{((\mathrm{F}(r)))}\,,\frac{d^{n-1}y}{dx^{n-1}}=$$

$$\Sigma(hx+k)^{r-n+1}\chi(r,x)\frac{h^{n-1}r(r-1)\ldots(r-n+2)}{((\mathrm{F}(r)))}\,,\ \frac{d^ny}{dx^n}=$$

$$\Sigma(hx+k)^{r-n}\chi(r,x)\frac{h^nr(r-1)\ldots(r-n+1)}{((\mathrm{F}(r)))}+$$

$$h^{n-1}\Sigma(hx+k)^{r-n+1}\frac{d\chi(r\,,x)}{dx}\,\frac{r(r-1)\ldots(r-n+2)}{((\mathrm{F}(r)))}\,;$$

factisque substitutionibus in (g^{VIII}),

$$\Sigma\,\frac{(hx+k)^{r-n}\chi(r\,,x)\,\mathrm{F}(r)}{((\mathrm{F}(r)))}+$$

$$h^{n-1}\Sigma(hx+k)^{r-n+1}\frac{d\chi(r\,,x)}{dx}\,\frac{r(r-1)\ldots(r-n+2)}{((\mathrm{F}(r)))}=f(x):$$

et quoniam $(hx+k)^{r-n}$, $\chi(r\,,x)$ haud evadunt infinitae quum loco r adhibentur valores ii, qui aequationi $\mathrm{F}(r)=0$ satisfaciunt, iccirco

$$\Sigma\frac{(hx+k)^{r-n}\chi(r\,,x)\,\mathrm{F}(r)}{((\mathrm{F}(r)))}=0\,,$$

et consequenter

$$h^{n-1}\,\Sigma\,(hx+k)^{r-n+1}\,\frac{d\chi(r,x)}{dx}\,\frac{r(r-1)\ldots(r-n+2)}{((\mathrm{F}(r)))}=f(x)\ldots(b^{\mathrm{VI}}).$$

Satisfiet igitur per (b^{IV}) aequationi (g^{VIII}), modo

$\chi(r, x)$ determinetur ita, ut (b^{v}), (b^{vi}) simul expleantur: atqui (69. 5.°)

$$\mathcal{E}\frac{r(r-1)\ldots(r-m+1)}{((\mathrm{F}(r)))}=0, \quad h^n\mathcal{E}\frac{r(r-1)\ldots(r-n+2)}{((\mathrm{F}(r)))}=1;$$

explebuntur itaque (b^{v}), (b^{vi}) sumendo

$$(hx+k)^{r-n+1}\frac{d\chi(r, x)}{dx}=hf(x),$$

ex qua (151) $\chi(r, x)=h\int_{x_0}^{x}(hz+k)^{n-r-1}f(z)dz+\varphi(r)$.

Hinc

$$y=\mathcal{E}\frac{(hx+k)^r\varphi(r)}{((\mathrm{F}(r)))}+h\,\mathcal{E}\frac{(hx+k)^r\int_{x_0}^{x}(hz+k)^{n-r-1}f(z)dz}{((\mathrm{F}(r)))}\ldots(g_3),$$

integrale completum (187) aequationis (g^{viii})

Si ponitur $f(x)=0$, ut habeamus

$$\left.\begin{aligned}&\frac{d^ny}{dx^n}+\frac{a_1}{hx+k}\frac{d^{n-1}y}{dx^{n-1}}+\frac{a_2}{(hx+k)^2}\frac{d^{n-2}y}{dx^{n-2}}+\ldots\\&\quad+\frac{a_{n-1}}{(hx+k)^{n-1}}\frac{dy}{dx}+\frac{a_ny}{(hx+k)^n}=0\end{aligned}\right\}(g^{ix}),$$

vertetur (g_3) in

$$y=\mathcal{E}\frac{(hx+k)^r\varphi(r)}{((\mathrm{F}(r)))}\ldots(g_4).$$

Exempla.

I.°

$$\frac{d^3y}{dx^3}-\frac{3}{x}\frac{d^2y}{dx^2}+\frac{6}{x^2}\frac{dy}{dx}-\frac{6}{x^3}y=\frac{1}{x^2}:$$

erunt $n=3$, $a_1=-3$, $a_2=6$, $a_3=-6$, $h=1$, $k=0$, $F(r)=r(r-1)(r-2)-3r(r-1)+6r-6=(r-1)(r-2)(r-3)$, $f(x)=\frac{1}{x^2}$; ideoque

$$y=\mathcal{E}\frac{x^r\varphi(r)}{(((r-1)(r-2)(r-3)))}+\mathcal{E}\frac{\int_{x_0}^{x} x^r z^{-r}dz}{(((r-1)(r-2)(r-3)))}.$$

Sed

$$\mathcal{E}\frac{x^r\varphi(r)}{(((r-1)(r-2)(r-3)))}=\frac{x\varphi(1)}{2}-x^2\varphi(2)+\frac{x^3\varphi(3)}{2},$$

et (136. 2.°)

$$\mathcal{E}\frac{\int_{x_0}^{x} x^r z^{-r}dz}{(((r-1)(r-2)(r-3)))}=\int_{x_0}^{x}\left(\frac{xz^{-1}}{2}-x^2z^{-2}+\frac{x^3z^{-3}}{2}\right)dz=$$

$$\frac{x}{2}\,L\left(\frac{x}{x_0}\right)+\frac{3x}{4}-\frac{x^2}{x_0}+\frac{x^3}{4x_0};$$

proinde

$$y=\frac{x}{2}\,L\left(\frac{x}{x_0}\right)+C_1x+\left(C_2-\frac{1}{x_0}\right)x^2+\left(C_3+\frac{1}{4x_0}\right)x^3.$$

II.°

$$\frac{d^2y}{dx^2}-\frac{1}{x+1}\,\frac{dy}{dx}+\frac{y}{(x+1)^2}=f(x):$$

erunt $n=2$, $h=1$, $k=1$, $a_1=-1$, $a_2=1$, $F(r)=r(r-1)-r+1=(r-1)^2$; proinde

$$y=\mathcal{E}\frac{(x+1)^r\varphi(r)}{(([r-1]^2))}+\mathcal{E}\frac{(x+1)^r\int_{x_0}^{x}(z+1)^{1-r}f(z)dz}{(([r-1]^2))}.$$

Sunt autem

$$\Sigma\frac{(x+1)^r\varphi(r)}{(([r-1]^2))}=\frac{d[(x+1)^{1+\theta}\varphi(1+\theta)]}{d\theta}=[\varphi(1)L(x+1)+\varphi'(1)](x+1),$$

$$\Sigma\frac{(x+1)^r\int_{x_0}^{x}(z+1)^{1-r}f(z)dz}{(((r-1)^2))}=$$

$$\int_{x_0}^{x}\frac{d[(x+1)^{1+\theta}(z+1)^{-\theta}f(z)dz}{d\theta}=(x+1)\int_{x_0}^{x}L(\frac{x+1}{z+1})f(z)dz:$$

igitur

$$y=(x+1)[C_1L(x+1)+C_2+\int_{x_0}^{x}L(\frac{x+1}{z+1})f(z)dz].$$

189. Data aequatione

$$\left.\begin{array}{l}\frac{d^ny}{dx^n}+a_1\frac{d^{n-1}y}{dx^{n-1}}+\ldots+a_{n-1}\frac{dy}{dx}+a_ny+\\ b\frac{d^nz}{dx^n}+b_1\frac{d^{n-1}z}{dx^{n-1}}+\ldots+b_{n-1}\frac{dz}{dx}+b_nz=f(x)\end{array}\right\}(g^1)$$

inter variabiles x, y, z, sit χ functio indeterminata quantitatis x: et facto

$$\left.\begin{array}{l}b\frac{d^nz}{dx^n}+b_1\frac{d^{n-1}z}{dx^{n-1}}+\ldots+b_{n-1}\frac{dz}{dx}+b_nz=\chi,\\ \text{erit}\\ \frac{d^ny}{dx^n}+a_1\frac{d^{n-1}y}{dx^{n-1}}+\ldots+a_{n-1}\frac{dy}{dx}+a_ny=f(x)-\chi.\end{array}\right\}(q)$$

Jam si binae (q) integrantur (187), altera suppeditabit $z=F(x, \chi)$, altera $y=F_1(x, \chi)$, per quas ma-

nifeste explebitur (g^{x}); huic nempe aequationi satisfacient infinitae numero lineae duplici praeditae curvedine.

Quod si habenda esset (g^{x}) pro aequatione ad superficiem curvam, facta prius $z=0$, dein $y=0$, exsurgerent

$$\left.\begin{aligned} &\frac{d^n y}{dx^n}+a_1\frac{d^{n-1}y}{dx^{n-1}}+\ldots+a_{n-1}\frac{dy}{dx}+a_n y=f(x),\\ &b\frac{d^n z}{dx^n}+b_1\frac{d^{n-1}z}{dx^{n-1}}+\ldots+b_{n-1}\frac{dz}{dx}+b_n z=f(x);\end{aligned}\right\}(q')$$

quibus integratis (187), aequationes inde prodeuntes ad communes superficiei curvae et planorum XOY, XOZ intersectiones pertinerent.

In aequationibus (g^{VI}),... (g^{X}) quantitates y, z, ... dy, d^2y, ... dz, d^2z, ... neque primum praetergrediuntur gradum, neque exsistunt invicem vel multiplicatae vel divisae: aequationes ejusmodi dicuntur *lineares*.

190. Datis

$$\left.\begin{aligned} &(a_1y+b_1z)dx^2+(h_1dy+i_1dz)dx+k_1d^2y+m_1d^2z=X_1dx^2,\\ &(a_2y+b_2z)dx^2+(h_2dy+i_2dz)dx+k_2d^2y+m_2d^2z=X_2dx^2,\end{aligned}\right\}(g^{XI})$$

inveniendi sint ejusmodi valores y, z expressi per x, quibus expleantur ambae (g^{XI}); a_1, ... m_1, a_2, ... m_2 denotant quantitates constantes, X_1 et X_2 functiones x. Pone

$$dy=udx, \quad dz=vdx;$$

mutabuntur (g^{XI}) in

$$\left.\begin{aligned} &(a_1y+b_1z+h_1u+i_1v)dx+k_1du+m_1dv=X_1dx,\\ &(a_2y+b_2z+h_2u+i_2v)dx+k_2du+m_2dv=X_2dx.\end{aligned}\right\}(p)$$

Ab his eliminata prius du, dein dv, et factis compendii causa

$$\left.\begin{array}{l}
\dfrac{a_1k_2-a_2k_1}{m_1k_2-m_2k_1}=A_1,\ \dfrac{b_1k_2-b_2k_1}{m_1k_2-m_2k_1}=B_1,\ \dfrac{h_1k_2-h_2k_1}{m_1k_2-m_2k_1}=H_1,\\
\qquad\qquad\qquad \dfrac{i_1k_2-i_2k_1}{m_1k_2-m_2k_1}=K_1,\ \dfrac{X_1k_2-X_2k_1}{m_1k_2-m_2k_1}=X',\\
\dfrac{a_2m_1-a_1m_2}{m_1k_2-m_2k_1}=A_2,\ \dfrac{b_2m_1-b_1m_2}{m_1k_2-m_2k_1}=B_2,\ \dfrac{h_2m_1-h_1m_2}{m_1k_2-m_2k_1}=H_2,\\
\qquad\qquad\qquad \dfrac{i_2m_1-i_1m_2}{m_1k_2-m_2k_1}=K_2,\ \dfrac{X_2m_1-X_1m_2}{m_1k_2-m_2k_1}=X'',
\end{array}\right\}(p')$$

prodibunt

$$\left.\begin{array}{l}
(A_1y+B_1z+H_1u+K_1v)dx+dv=X'dx\\
(A_2y+B_2z+H_2u+K_2v)dx+du=X''dx,\\
\text{quibus addendae}\\
dy-udx=0,\ dz-vdx=0.
\end{array}\right\}(p'')$$

Multiplica (175 : 178) secundam (p'') per α, tertiam per ν, quartam per ω, et quae proveniunt facta adde primae; exsistet

$$\left.\begin{array}{l}
(K_1+K_2\alpha-\omega)\left(\dfrac{A_1+A_2\alpha}{K_1+K_2\alpha-\omega}y+\dfrac{B_1+B_2\alpha}{K_1+K_2\alpha-\omega}z+\right.\\
\left.\dfrac{H_1+H_2\alpha}{K_1+K_2\alpha-\omega}u+v\right)dx+\nu dy+\omega dz+\alpha du+dv=(X'+\alpha X'')\,dx
\end{array}\right\}(p''').$$

Ad indeterminatas α, ν, ω definiendas fac

$$\frac{A_1+A_2\alpha}{K_1+K_2\alpha-\omega}=\nu,\ \frac{B_1+B_2\alpha}{K_1+K_2\alpha-\omega}=\omega,\ \frac{H_1+H_2\alpha}{K_1+K_2\alpha+\omega}=\alpha;$$

emerget plerumque aequatio sexti gradus in ordine ad α; hinc sex valores α; et duobus, qui minus opportuni censentur, omissis, retine caeteros α', α'', α''', α^{IV}; exinde inferes ν', ν'', ν''', ν^{IV}, nec non ω', ω'', ω''', ω^{IV}; sicque habebis ex (p''')

$(K_1 K_2 \alpha' - \omega')(\nu' y + \omega' z + \alpha' u + v) dx + d(\nu' y + \omega' z + \alpha' u + v) = (X' + \alpha' X'') dx$; $(K_1 + K_2 \alpha'' - \omega'')(\nu'' y + \omega'' z + \alpha'' u + v) dx + d(\nu'' y + \omega'' z + \alpha'' u + v) = (X' + \alpha'' X'') dx$, $(K_1 + K_2 \alpha''' - \omega''')(\nu''' y + \omega''' z + \alpha''' u + v) dx + d(\nu''' y + \omega''' z + \alpha''' u + v) = (X' + \alpha''' X'') dx$, $(K_1 + K_2 \alpha^{IV} - \omega^{IV})(\nu^{IV} y + \omega^{IV} z + \alpha^{IV} u + v) dx + d(\nu^{IV} y + \omega^{IV} z + \alpha^{IV} u + v) = (X' + \alpha^{IV} X'') dx$;

quae suppeditabunt (156 : I.°)

$$\left.\begin{array}{l} \nu' y + \omega' z + \alpha' u + v = F_1(x), \quad \nu'' y + \omega'' z + \alpha'' u + v = F_2(x), \\ \nu''' y + \omega''' z + \alpha''' u + v = F_3(x), \nu^{IV} y + \omega^{IV} z + \alpha^{IV} u + v = F_4(x). \end{array}\right\} (p^{IV})$$

Eliminata ab (p^{IV}) prius z, u, v, dein y, u, v; obvenient y, z expressae per x.

191. Si $h_1 = 0$, $i_1 = 0$, $h_2 = 0$, $i_2 = 0$, et X_1, X_2 exhibent quantitates constantes r_1, r_2 ut immutentur (g^{XI}) in

$$\left.\begin{array}{l} (a_1 y + b_1 z) dx^2 + k_1 d^2 y + m_1 d^2 z = r_1 dx^2, \\ (a_2 y + b_2 z) dx^2 + k_2 d^2 y + m_2 d^2 z = r_2 dx^2, \end{array}\right\} (g^{XII})$$

multo facilius integratio succedet hac ratione : elimina ab (g^{XII}) prius $d^2 z$, dein $d^2 y$; et factis brevitatis gratia

$$\left.\begin{array}{l} \dfrac{a_1 m_2 - a_2 m_1}{k_1 m_2 - k_2 m_1} = A_1, \quad \dfrac{b_1 m_2 - b_2 m_1}{k_1 m_2 - k_2 m_1} = B_1, \quad \dfrac{r_1 m_2 - r_2 m_1}{k_1 m_2 - k_2 m_1} = R_1 \\ \dfrac{a_2 k_1 - a_1 k_2}{k_1 m_2 - k_2 m_1} = A_2, \quad \dfrac{b_2 k_1 - b_1 k_2}{k_1 m_2 - k_2 m_1} = B_2, \quad \dfrac{r_2 k_1 - r_1 k_2}{k_1 m_2 - k_2 m_1} = R_2, \end{array}\right\} (p^{V})$$

habebis

$$\left.\begin{array}{l} (A_1 y + B_1 z) dx^2 + d^2 y = R_1 dx^2, \\ (A_2 y + B_2 z) dx^2 + d^2 z = R_2 dx^2. \end{array}\right\} (p^{VI})$$

Secundam (p^{VI}) multiplicatam per α adde primae; erit

$$\left.\begin{array}{l}(A_1+A_2\alpha)(y+\dfrac{B_1+B_2\alpha}{A_1+A_2\alpha}z)dx^2+d^2y+\alpha d^2z= \\ \qquad (R_1+R_2\alpha)dx^2.\end{array}\right\}(p^{\text{VII}})$$

Fac

$$\left.\begin{array}{l}\dfrac{B_1+B_2\alpha}{A_1+A_2\alpha}=\alpha;\ \text{prodibit} \\ \alpha=\dfrac{B_2-A_1\pm\sqrt{[4A_2B_1+(A_1-B_2)^2]}}{2A_2};\end{array}\right\}(p^{\text{VIII}})$$

et expressis per α', α'' valoribus α, positisque

$$\left.\begin{array}{l}y+\alpha'z-\dfrac{R_1+R_2\alpha'}{A_1+A_2\alpha'}=v', \\ y+\alpha''z-\dfrac{R_1+R_2\alpha''}{A_1+A_2\alpha''}=v'',\end{array}\right\}(p^{\text{IX}})$$

aequatio (p^{VII}) dabit

$$\frac{d^2v'}{dx^2}=-(A_1+A_2\alpha')v',\ \frac{d^2v''}{dx^2}=-(A_1+A_2\alpha'')v'';$$

unde (182 : 1.°)

$$dx=\frac{dv'}{v'\sqrt{[-(A_1+A_2\alpha')]}},\ dx=\frac{dv''}{v''\sqrt{[-(A_1+A_2\alpha'')]}};$$

et consequenter

$$x=\frac{1}{\sqrt{[-(A_1+A_2\alpha')]}}L(\frac{v'}{C_1}),\ x=\frac{1}{\sqrt{[-(A_1+A_2\alpha'')]}}L(\frac{v''}{C_2}).$$

Hinc ob (p^{IX})

$$\left.\begin{array}{l}y+\alpha'z-\dfrac{R_1+R_2\alpha'}{A_1+A_2\alpha'}=C_1e^{x\sqrt{[-(A_1+A_2\alpha')]}}, \\ y+\alpha''z-\dfrac{R_1+R_2\alpha''}{A_1+A_2\alpha''}=C_2e^{x\sqrt{[-(A_1+A_2\alpha'')]}};\end{array}\right\}(p^{\text{X}})$$

e quibus y et z expressae per x. Datis v. gr.

$$(17y+88z)dx^2-8d^2y-7d^2z=59dx^2\ ,$$
$$(11y+52z)dx^2-5d^2y-4d^2z=35dx^2\ ,$$

erunt (g^{XII})

$a_1=17$, $b_1=88$, $k_1=-8$, $m_1=-7$, $r_1=59$, $a_2=11$, $b_2=52$, $k_2=-5$, $m_2=-4$, $r_2=35$; proinde (p^{V})

$A_1=-3$, $B_1=-4$, $R_1=-3$, $A_2=1$, $B_2=-8$, $R_2=-5$; insuper (p^{VIII})

$$\alpha'=-1\ ,\ \alpha''=-4:$$

ergo (p^{X})

$$y-z+\tfrac{1}{2}=C_1e^{2x}\ ,\ y-4z+\frac{17}{7}=C_2e^{x\sqrt{7}}\ ;$$

ideoque

$$y=\frac{4C_1e^{2x}-C_2e^{x\sqrt{7}}}{3}+\frac{1}{7}\ ,\ z=\frac{C_1e^{2x}-C_2e^{x\sqrt{7}}}{3}+\frac{9}{14}.$$

Si $4A_2B_1+(A_1-B_2)^2=0$, et consequenter $\alpha'=\alpha''$, binae (p^{X}) recident in unam; verum facto

$$C_1e^{x\sqrt{[-(A_1+A_2\alpha')]}}+\frac{R_1+R_2\alpha'}{A_1+A_2\alpha'}=\chi(x)\ \text{ut sit}$$

$y=\chi(x)-\alpha'z$, et adhibitis substitutionibus in prima (p^{VI}), obveniet aequatio

$$\frac{d^2z}{dx^2}+\frac{A_1\alpha'-B_1}{\alpha'}z=\frac{A_1\chi(x)+\chi''(x)-R_1}{\alpha'}\ .$$

192. De his satis; nunc redeo ad lineares aequationes (g^{VI}), ... (g^{IX}) ut, datis valoribus quos recipiunt

$$y\ ,\ \frac{dy}{dx}\ ,\ \frac{d^2y}{dx^2}\ ,\ldots\frac{d^{n-1}y}{dx^{n-1}}\ldots(t)$$

quum loco x adhibetur peculiaris valor x_0, ex iis valoribus determinem arbitrarias functiones quae in (g_1), ... (g_k) ingrediuntur.

Veniant 1.° considerandae (g^{VI}) et (g_1), constituantque valores illi progressionem geometricam

$$\alpha^0(=1), \alpha^1, \alpha^2, \alpha^3, \ldots \alpha^{n-1},$$

quicumque caeteroquin sit α : cum, assumpto $m<n$, habeamus (69. 5.°)

$$\mathcal{E}\frac{de^{rx}\int_{x_0}^{x} e^{-rx}f(x)dx}{dx}\frac{1}{((F(r)))}=$$

$$\mathcal{E}\left[re^{rx}\int_{x_0}^{x} e^{-rx}f(x)dx+f(x)\right]\frac{1}{((F(r)))}=$$

$$\mathcal{E}\frac{re^{rx}\int_{x_0}^{x} e^{-rx}f(x)dx}{((F(r)))}, \mathcal{E}\frac{d^2e^{rx}\int_{x_0}^{x} e^{-rx}f(x)dx}{dx^2}\frac{1}{((F(r)))}=$$

$$\mathcal{E}\frac{dre^{rx}\int_{x_0}^{x} e^{-rx}f(x)dx}{dx}\frac{1}{((F(r)))}=\mathcal{E}\frac{r^2e^{rx}\int_{x_0}^{x} e^{rx}f(x)dx}{((F(r)))},$$

et caet. . . . ,

$$\mathcal{E}\frac{d^me^{rx}\int_{x_0}^{x} e^{-rx}f(x)dx}{dx^m}\frac{1}{((F(r)))}=\mathcal{E}\frac{r^me^{rx}\int_{x_0}^{x} e^{-rx}f(x)dx}{((F(r)))},$$

et consequenter

$$\frac{d^my}{dx^m}=\mathcal{E}\frac{r^me^{rx}\varphi(r)}{((\mathrm{F}(r)))}+\mathcal{E}\frac{d^me^{rx}\int_{x_0}^{x}e^{-rx}f(x)dx}{dx^m}\cdot\frac{1}{((\mathrm{F}(r)))}=$$

$$\mathcal{E}\frac{r^me^{rx}\varphi(r)}{((\mathrm{F}(r)))}+\mathcal{E}\frac{r^m\int_{x_0}^{x}e^{r(x-z)}f(z)dz}{((\mathrm{F}(r)))};$$

iccirco

$$\mathcal{E}\frac{r^me^{rx_0}\varphi(r)}{((\mathrm{F}(r)))}+\mathcal{E}\frac{r^m\int_{x_0}^{x}e^{r(x_0-z)}f(z)dz}{((\mathrm{F}(r)))}=\alpha^m.$$

Sed (127 : 69. 5.°)

$$\mathcal{E}\frac{r^m\int_{x_0}^{x_0}e^{r(x_0-z)}f(z)dz}{((\mathrm{F}(r)))}=0;$$

insuper (67. 1.°)

$$\alpha^m=\mathcal{E}\frac{r^m}{((r-\alpha))}:$$

igitur

$$\mathcal{E}\frac{r^me^{rx_0}\varphi(r)}{((\mathrm{F}(r)))}-\mathcal{E}\frac{r^m}{((r-\alpha))}=0,$$

seu

$$\mathcal{E}\frac{r^m(r-\alpha)e^{rx_0}\varphi(r)-\mathrm{F}(r)}{(((r-\alpha)\mathrm{F}(r)))}=0\ldots(t');$$

sic nempe determinanda est functio arbitraria $\varphi(r)$ ut expleatur (t'). Jamvero (69. 5°.)

$$\mathcal{E}\frac{r^m}{((r-\alpha)F(r))}=0;$$

explebitur itaque (t') ponendo

$$(r-\alpha)e^{rx_0}\varphi(r)-F(r)=C.$$

Haec autem, facto $r=\alpha$, praebet $C=-F(\alpha)$, ideoque

$$(r-\alpha)e^{rx_0}\varphi(r)-F(r)=-F(\alpha):$$

ergo

$$\varphi(r)=\frac{F(r)-F(\alpha)}{(r-\alpha)e^{rx_0}}\ldots(t'');$$

et adhibita substitutione in (g_1),

$$y=\mathcal{E}\frac{F(r)-F(\alpha)}{r-\alpha}e^{-rx_0}\frac{e^{rx}}{((F(r)))}+\mathcal{E}\frac{\int_{x_0}^{x}e^{r(x-z)}f(z)dz}{((F(r)))}\ldots(t''').$$

Quoniam

$$\frac{F(r)-F(\alpha)}{r-\alpha}=\frac{r^n-\alpha^n+a_1(r^{n-1}-\alpha^{n-1})+\ldots+a_{n-1}(r-\alpha)}{r-\alpha},$$

poterit secundum membrum (t''') sic exprimi

$$P+Q\alpha+R\alpha^2+\ldots+T\alpha^{n-3}+U\alpha^{n-2}+V\alpha^{n-1}:$$

proinde

$$y=P\alpha^0+Q\alpha'+R\alpha^2+\ldots+T\alpha^{n-3}+U\alpha^{n-2}+V\alpha^{n-1};$$

P, Q, ... V non continent α, suntque ejusmodi ut peculiari valori x_0 manifeste respondeant

$$\left.\begin{array}{l}
P=1,\quad Q=0,\quad R=0,\ldots\; V=0,\\
\dfrac{dP}{dx}=0,\quad \dfrac{dQ}{dx}=1,\quad \dfrac{dR}{dx}=0,\ldots\; \dfrac{dV}{dx}=0,\\
\dfrac{d^2P}{dx^2}=0,\quad \dfrac{d^2Q}{dx^2}=0,\quad \dfrac{d^2R}{dx^2}=1,\ldots\; \dfrac{d^2V}{dx^2}=0,\\
\text{et caet.}\ldots\\
\dfrac{d^{n-1}P}{dx^{n-1}}=0,\dfrac{d^{n-1}Q}{dx^{n-1}}=0,\dfrac{d^{n-1}R}{dx^{n-1}}=0,\ldots,\dfrac{d^{n-1}V}{dx^{n-1}}=1.
\end{array}\right\}(t^{IV}).$$

Ponamus nunc valores (t) respondentes peculiari valori x_0 haud constituere progressionem geometricam, sed esse quantitates quascumque

$$\alpha_0\,,\ \alpha_1\,,\ \alpha_2\,,\ldots\alpha_{n-1}\ldots(t^V):$$

in (g_1) erit determinanda $\varphi(r)$ ita, ut, adhibito x_0 pro x, vertantur (t) in (t^V). Jamvero assumpta

$$y=P\alpha_0+Q\alpha_1+R\alpha_2+\ldots+V\alpha_{n-1},$$

reipsa ob (t^{IV}) vertuntur (t) in (t^V); obtinentur ergo $\varphi(r)$ et y ab (t'') et (t''') immutando

$$\alpha^0\,,\ \alpha^1\,,\ \alpha^2\ldots\alpha^{n-1}\ \text{in}\ \alpha_0\,,\ \alpha_1\,,\ \alpha_2\ldots\alpha_{n-1}$$

posteaquam $F(r)-F(\alpha)$ divisa est per $r-\alpha$.

Quod pertinet 2.° ad (g^{VII}) et (g_2), simili modo habentur $\varphi(r)$ et y ab (t'') et (t'''), facta $f(z)=0$.

Exempla.

I.° Debeat integrari aequatio (187. II.°)

$$\frac{d^2y}{dx^2}+ay=f(x)$$

ita, ut peculiari valori $x_0=0$ respondeant $y=\alpha_0$,

$\frac{dy}{dx}=\alpha_1$: erunt $n=2$, $a_1=0$, $a_2=a$, $F(r)=r^2+a$,
$\frac{F(r)-F(\alpha)}{r-\alpha}=\frac{r^2-\alpha^2}{r-\alpha}=r\alpha^0+\alpha'$, $\varphi(r)=r\alpha_0+\alpha_1$;
ideoque

$$y=\Sigma(r\alpha_0+\alpha_1)\frac{e^{rx}}{((r^2+a))}+\Sigma\frac{\int_0^x e^{r(x-z)}f(z)dz}{((r^2+a))}.$$

Sed

$$\Sigma(r\alpha_0+\alpha_1)\frac{e^{rx}}{((r^2+a))}=\frac{\alpha_0}{2}(e^{x\sqrt{a}\sqrt{-1}}+e^{-x\sqrt{a}\sqrt{-1}})+$$

$$\frac{\alpha_1}{2}(e^{x\sqrt{a}\sqrt{-1}}-e^{-x\sqrt{a}\sqrt{-1}})=\alpha_0\cos x\sqrt{a}+$$

$$\frac{\alpha_1}{\sqrt{a}}\sin x\sqrt{a}\text{, et }\Sigma\frac{\int_0^x e^{r(x-z)}f(z)dz}{((r^2+a))}=$$

$$\frac{1}{2\sqrt{a}\sqrt{-1}}\int_0^x[e^{(x-z)\sqrt{a}\sqrt{-1}}-e^{-(x-z)\sqrt{a}\sqrt{-1}}]f(z)dz=$$

$$\frac{1}{\sqrt{a}}\int_0^x\sin(x-z)\sqrt{a}\,.\,f(z)dz;$$

igitur

$$y=\alpha_0\cos x\sqrt{a}+\frac{\alpha_1}{\sqrt{a}}\sin x\sqrt{a}+\frac{1}{\sqrt{a}}\int_0^x\sin(x-z)\sqrt{a}\,.f(z)dz.$$

II.° Debeat quoque integrari aequatio

$$\frac{d^2y}{dx^2}+a^2y=0$$

ita, ut peculiari valori x_0 respondeant $y=1$, $\frac{dy}{dx}=0$: erunt $n=2$, $a_1=0$, $a_2=a^2$, $f(x)=0$, $F(r)=r^2+a^2$, $\frac{F(r)-F(\alpha)}{r-\alpha}=r\alpha^0+\alpha^1$, $\varphi(r)=r\alpha_0+\alpha_1=r$; hinc

$$y=\mathcal{E}\frac{re^{rx}}{((r^2+a^2))}=\frac{e^{ax\sqrt{-1}}+e^{-ax\sqrt{-1}}}{2}=\cos ax.$$

Veniant 3.° considerandae (g^{VIII}) et (g_2), sintque

$$\alpha^0(=1),\ \frac{h\alpha}{hx_0+k},\ \frac{h^2\alpha(\alpha-1)}{(hx_0+k)^2},\ \ldots\ \frac{h^{n-1}\alpha(\alpha-1)\ldots(\alpha-n+2)}{(hx_0+k)^{n-1}}$$

valores (t) quum adhibetur x_0 pro x. Quoniam (69. 5.°), assumpto $m<n$,

$$\mathcal{E}\frac{d(hx+k)^r\int_{x_0}^{x}(hx+k)^{n-r-1}f(x)dx}{dx}\,\frac{1}{((F(r)))}=$$

$$\mathcal{E}[hr(hx+k)^{r-1}\int_{x_0}^{x}(hx+k)^{n-r-1}f(x)dx+$$

$$(hx+k)^{n-1}f(x)]\frac{1}{((F(r)))}=$$

$$h\mathcal{E}\frac{r(hx+k)^{r-1}\int_{x_0}^{x}(hx+k)^{n-r-1}f(x)dx}{((F(r)))},$$

$$\mathcal{E}\frac{d^2(hx+k)^r\int_{x_0}^{x}(hx+k)^{n-r-1}f(x)dx}{dx^2}\,\frac{1}{((F(r)))}=$$

$$h^2 \mathcal{E} \frac{r(r-1)(hx+k)^{r-2}\int_{x_0}^{x}(hx+k)^{n-r-1}f(x)dx}{((F(r)))},$$

et caet....,

$$\mathcal{E}\frac{d^m(hx+k)^r\int_{x_0}^{x}(hx+k)^{n-r-1}f(x)dx}{dx^m}\frac{1}{((F(r)))}=$$

$$=h^m\mathcal{E}\frac{r(r-1)..(r-m+1)(hx+k)^{r-m}\int_{x_0}^{x}(hx+k)^{n-r-1}f(x)dx}{((F(r)))};$$

et consequenter

$$\frac{d^m y}{dx^m}=h^m\mathcal{E}\frac{r(r-1)\ldots(r-m+1)(hx+k)^{r-m}\varphi(r)}{((F(r)))}+$$

$$h^{m+1}\mathcal{E}\frac{r(r-1)...(r-m+1)(hx+k)^{r-m}\int_{x_0}^{x}(hz+k)^{n-r-1}f(z)dz}{((F(r)))};$$

iccirco

$$h^m\mathcal{E}\frac{r(r-1)\ldots(r-m+1)(hx_0+k)^{r-m}}{((F(r)))}[\varphi(r)+$$

$$h\int_{x_0}^{x_0}(hz+k)^{n-r-1}f(z)dz]=\frac{h^m\alpha(\alpha-1)...(\alpha-m+1)}{(hx_0+k)^m}.$$

Sed (127)

$$\int_{x_0}^{x_0}(hz+k)^{n-r-1}f(z)dz=0;$$

et (67. 1.°)

$$\alpha(\alpha-1)\ldots(\alpha-m+1)=\Sigma\frac{r(r-1)\ldots(r-m+1)}{((r-\alpha))};$$

igitur

$$\Sigma\frac{r(r-1)\ldots(r-m+1)(hx_0+k)^r\varphi(r)}{((F(r)))}-\Sigma\frac{r(r-1)\ldots(r-m+1)}{((r-\alpha))}=0;$$

seu

$$\Sigma\frac{r(r-1)\ldots(r-m+1)}{(((r-\alpha)F(r)))}[(r-\alpha)(hx_0+k)^r\varphi(r)]=0\ldots(t^{VI}):$$

sic nempe determinanda est $\varphi(r)$, ut expleatur (t^{VI}). Jamvero (69. 5.°)

$$\Sigma\frac{r(r-1)\ldots(r-m+1)}{(((r-\alpha)F(r)))}=0:$$

explebitur itaque (t^{VI}) ponendo

$$(r-\alpha)(hx_0+k)^r\varphi(r)-F(r)=C.$$

Haec autem, facto $r=\alpha$, praebet $C=-F(\alpha)$; ideoque

$$(r-\alpha)(hx_0+k)^r\varphi(r)-F(r)=-F(\alpha);$$

ergo

$$\varphi(r)=\frac{F(r)-F(\alpha)}{r-\alpha}(hx_0+k)^{-r}\ldots(t^{VII}):$$

et adhibita substitutione in (g_3),

$$\left.\begin{aligned}y=\Sigma\frac{F(r)-F(\alpha)}{r-\alpha}\left(\frac{hx+k}{hx_0+k}\right)^r\frac{1}{((F(r)))}+\\ h\Sigma\frac{(hx+k)^r\int_{x_0}^{x}(hz+k)^{n-r-1}f(z)dz}{((F(r)))}.\end{aligned}\right\}(t^{VIII})$$

Evolvatur fractio

$$\frac{F(r)-F(\alpha)}{r-\alpha}$$

ita, ut procedant termini quoad α juxta

$$\alpha^0\,,\ \alpha^1,\ \alpha(\alpha-1)\,,\ldots\ \alpha(\alpha-1)\ldots(\alpha-n+2):$$

poterit (t^{VIII}) exprimi in hunc modum

$$y=P\alpha^0+Q\alpha^1+R\alpha(\alpha-1)+\ldots+V\alpha(\alpha-1)\ldots(\alpha-n+2)\,;$$

P, Q, ... V non continent α, suntque ejusmodi ut peculiari valori x_0 manifeste respondeant

$$\left.\begin{array}{l}
P=1,\quad Q=0,\quad R=0,\ldots\quad V=0,\\
\dfrac{dP}{dx}=0,\quad \dfrac{dQ}{dx}=\dfrac{h}{hx_0+k},\ \dfrac{dR}{dx}=0,\ldots\ \dfrac{dV}{dx}=0,\\
\dfrac{d^2P}{dx^2}=0,\ \dfrac{d^2Q}{dx^2}=0,\ \dfrac{d^2R}{dx^2}=\dfrac{h^2}{(hx_0+k)^2},\ldots\ \dfrac{d^2V}{dx^2}=0,\\
\text{et caet.}\ \ldots,\\
\dfrac{d^{n-1}P}{dx^{n-1}}=0,\ \dfrac{d^{n-1}Q}{dx^{n-1}}=0,\ \dfrac{d^{n-1}R}{dx^{n-1}}=0,\ldots,\ \dfrac{d^{n-1}V}{dx^{n-1}}=\dfrac{h^{n-1}}{(hx_0+k)^{n-1}}
\end{array}\right\}(t^{\mathrm{IX}})$$

Valores (t) respondentes peculiari valori x_0 ponantur esse quantitates quaevis (t^{V}): in (g_2) erit determinanda $\varphi(r)$ ita, ut adhibito x_0 pro x vertantur (t) in (t^{V}). Atqui assumpta

$$y=P\alpha_0+Q\alpha_1\frac{hx_0+k}{h}+R\alpha_2\frac{(hx_0+k)^2}{h^2}+\ldots+V\alpha_{n-1}\frac{(hx_0+k)^{n-1}}{h^{n-1}},$$

reipsa ob (t^{IX}) vertuntur (t) in (t^{V}); obtinentur ergo $\varphi(r)$ et y ab (t^{VII}) et (t^{VIII}) immutando

$$\alpha^0\,,\ \alpha^1,\ \alpha(\alpha-1)\,,\ldots\ \alpha(\alpha-1)\ldots(\alpha-n+2)$$

in

$$\alpha_0\,,\ \alpha_1\left(\frac{hx_0+k}{h}\right),\ \alpha_2\left(\frac{hx_0+k}{h}\right)^2,\ \ldots\ \alpha_{n-1}\left(\frac{hx_0+k}{h}\right)^{n-1}$$

posteaquam facta est praefata fractionis evolutio.

Sit v. gr. integranda

$$\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}+\frac{y}{x^2}=f(x)$$

ita, ut peculiari valori $x_0=1$ respondeant $y=\alpha_0$, $\frac{dy}{dx}=\alpha_1$: erunt $n=2$, $h=1$, $k=0$, $a_1=1$, $a_2=1$, $F(r)=r(r-1)+r+1=r^2+1$, $\frac{F(r)-F(\alpha)}{r-\alpha}=\frac{r^2-\alpha^2}{r-\alpha}=r\alpha^0+\alpha^1$, $\varphi(r)=r\alpha_0+\alpha_1$, ideoque

$$y=\mathcal{E}\frac{(r\alpha_0+\alpha_1)x^r}{((r^2+1))}+\mathcal{E}\frac{x^r\int_1^x z^{1-r}f(z)dz}{((r^2+1))}.$$

Est autem

$$\mathcal{E}\frac{(r\alpha_0+\alpha_1)x^r}{((r^2+1))}=\frac{\alpha_0}{2}(x^{\sqrt{-1}}+x^{-\sqrt{-1}})+\frac{\alpha_1}{2\sqrt{-1}}(x^{\sqrt{-1}}-x^{-\sqrt{-1}}),$$

seu (162. 2.° ex p. 2.ª)

$$\mathcal{E}\frac{(r\alpha_0+\alpha_1)x^r}{((r^2+1))}=\alpha_0\cos\mathrm{L}(x)+\alpha_1\sin\mathrm{L}(x);$$

insuper

$$\mathcal{E}\frac{x^r\int_1^x z^{1-r}f(z)dz}{((r^2+1))}=\int_1^x\mathcal{E}\frac{z\left(\frac{x}{z}\right)^r f(z)dz}{((r^2+1))}=$$

$$\frac{1}{2\sqrt{-1}}\int_1^x\left[\left(\frac{x}{z}\right)^{\sqrt{-1}}-\left(\frac{x}{z}\right)^{-\sqrt{-1}}\right]zf(z)dz=$$

$$\int_1^x zf(z)\,sin\mathrm{L}\left(\frac{x}{z}\right)dz:$$

igitur

$$y=\alpha_0 cos\mathrm{L}(x)+\alpha_1 sin\mathrm{L}(x)+\int_1^x zf(z)sin\mathrm{L}\left(\frac{x}{z}\right)dz.$$

Quod spectat 4.° ad (g^{IX}) et (g_4), $\varphi(r)$ et γ habentur ut supra ab (t^{VII}) et (t^{VIII}), facta $f(z)=0$. Sic ubi detur integranda v. gr.

$$\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}+\frac{y}{x^2}=0$$

sub iisdem conditionibus ac in exemplo modo allato, erit

$$y=\alpha_0 cos\mathrm{L}(x)+\alpha_1 sin\mathrm{L}(x).$$

DE INTEGRATIONE DIFFERENTIALIUM AEQUATIONUM PER SERIES.

193. Sit aequatio primi ordinis

$$\frac{dy}{dx}=\mathrm{F}(x\,,\,y)\ldots(h)$$

integranda ita, ut peculiari valori $x=x_0$ respondeat $y=y_0$. Exhibeatur integrale per

$$y=\chi(x);$$

erit (32)

$$\chi(x+\delta)=\chi(x)+\delta\chi'(x)+\frac{\delta^2}{2}\chi''(x)+\frac{\delta^3}{2.3}\chi'''(x)+\ldots\ldots;$$

et quoniam

$$\left.\begin{array}{l}\chi'(x)=\frac{dy}{dx}=F(x,y),\ \chi''(x)=\frac{d^2y}{dx^2}=F'_x(x,y)+\\ F'_y(x,y)F(x,y)=f(x,y),\ \chi'''(x)=\frac{d^3y}{dx^3}=\\ f'_x(x,y)+f'_y(x,y)F(x,y)=\mathrm{f}(x,y),\\ \chi^{\mathrm{iv}}(x)=\frac{d^4y}{dx^4}=\mathrm{f}'_x(x,y)+\mathrm{f}'_y(x,y)F(x,y)=\psi(x,y),\\ \text{et caet.}\ \ldots,\end{array}\right\}(h')$$

iccirco

$$\chi(x+\delta)=y+\delta F(x,y)+\frac{\delta^2}{2}f(x,y)+\frac{\delta^3}{2.3}\mathrm{f}(x,y)+\ldots;$$

et adhibito x_0 pro x,

$$\chi(x_0+\delta)=y_0+\delta F(x_0,y_0)+\frac{\delta^2}{2}f(x_0,y_0)+\frac{\delta^3}{2.3}\mathrm{f}(x_0,y_0)+\ldots,$$

quae, facta $\delta=x-x_0$, praebebit $\chi(x)$ seu

$$\left.\begin{array}{c}y=y_0+(x-x_0)F(x_0,y_0)+\\ \frac{(x-x_0)^2}{2}f(x_0,y_0)+\frac{(x-x_0)^3}{2.3}\mathrm{f}(x_0,y_0)+\ldots\end{array}\right\}(h'').$$

Detur v. gr.

$$\frac{dy}{dx}=\frac{my}{1+x}:$$

ex (h') habemus

$$F(x_0,y_0)=\frac{my_0}{1+x_0},\ f(x_0,y_0)=\frac{m(m-1)y_0}{(1+x_0)^2},$$

$$\mathrm{f}(x_0,y_0)=\frac{m(m-1)(m-2)y_0}{(1+x_0)^3},\ \text{et caet.}\ \ldots;$$

igitur

$$y=y_0[1+m\frac{x-x_0}{1+x_0}+\frac{m(m-1)}{2}(\frac{x-x_0}{1+x_0})^2+$$
$$\frac{m(m-1)(m-2)}{2.3}(\frac{x-x_0}{1+x_0})^3+\ldots].$$

Si forte evenit ut sese offerat summa seriei (h''), assequemur haud dubie integrale accuratum sub forma finita; sic in allato exemplo est

$$y=y_0(1+\frac{x-x_0}{1+x_0})^m=y_0(\frac{1+x}{1+x_0})^m:$$

quod integrale obtinetur etiam ab (g_4. 188), factis $n=1$, $a_1=0$, $a_2=0$, ... $a_{n-1}=0$, $a_n=-m$, $h=1$, $k=1$, $F(r)=r-m$; prodibit enim

$$y=\mathcal{E}\frac{(x+1)^r\varphi(r)}{((r-m))}=(x+1)^m\varphi(m)=C(x+1)^m;$$

et adhibito x_0 pro x, obveniet $y_0=C(x_0+1)^m$, unde

$$C=\frac{y_0}{(x_0+1)^m},\ \text{ideoque}\ y=y_0(\frac{x+1}{x_0+1})^m.$$

Idipsum erueretur ex (156. I.°), factis $X=-\frac{m}{1+x}$, $X_1=0$; forent namque $z=(1+x)^m$, $s=C$; proinde $y=C(1+x)^m$, et caet. . . .

194. Potest etiam hoc pacto integrari (h) per series: sume

$$y=y_0+a_1(x-x_0)^\alpha+a_2(x-x_0)^{\alpha+1}+a_3(x-x_0)^{\alpha+2}+\ldots\ (h'''),$$

quae, facto compendii causa

$$x-x_0=z,\ \text{ideoque}\ x=z+x_0,\ dz=dx,$$

vertetur in

$$y=y_0+a_1z^{\alpha}+a_2z^{\alpha+1}+a_3z^{\alpha+2}+\ldots(h^{IV}),$$

unde

$$\frac{dy}{dx}=a_1\alpha z^{\alpha-1}+a_2(\alpha+1)z^{\alpha}+a_3(\alpha+2)z^{\alpha+1}+\ldots(h^{V}):$$

in (h) substitue valores x, y, $\frac{dy}{dx}$; erit

$$(h^{V})-F(z+x_0, (h^{IV}))=0\ldots(h^{VI});$$

ex qua ordinata quoad potentias crescentes quantitatis z eruentur α, a_1, a_2, ... Ut res declaretur exemplo, proponatur aequatio

$$dy-ydx-bx^m dx=0$$

sic integranda, ut valori $x=0$ respondeat $y=0$: erunt $x_0=0$, $y_0=0$, $F(x, y)=y+bx^m$; et

$$bz^m-a_1\alpha z^{\alpha-1}+[a_1-a_2(\alpha+1)]z^{\alpha}+[a_2-a_3(\alpha+2)]z^{\alpha+1}+\ldots=0;$$

quae cum valere debeat utcumque se habet z, evanescent coefficientes singuli potentiarum z; et quia b haud evanescit per se, ideo sic erit determinanda α ut b non sit considerandus seorsum ab alio quovis coefficiente. Jam si fiat $\alpha-1=m$, vertetur aequatio in

$$[b-a_1(m+1)]z^m+[a_1-a_2(m+2)]z^{m+1}+$$
$$[a_2-a_3(m+3)]z^{m+2}+\ldots=0,$$

eruntque

$$b-a_1(m+1)=0, \; a_1-a_2(m+2)=0,$$
$$a_2-a_3(m+3)=0, \text{ et caet.} \ldots$$

unde

$$a_1 = \frac{b}{m+1}, \; a_2 = \frac{b}{(m+1)(m+2)}, \; a_3 = \frac{b}{(m+1)(m+2)(m+3)}, \text{ et caet...}$$

et consequenter

$$y = \frac{b}{m+1} x^{m+1} \left(1 + \frac{x}{m+2} + \frac{x^2}{(m+2)(m+3)} + \ldots\right).$$

Facto $m=0$, prodibit (241 ex p. 1.[a])

$$y = b\left(x + \frac{x^2}{2} + \frac{x^3}{2.3} + \ldots\right) = b(e^x - 1).$$

195. Data aequatione secundi ordinis

$$\frac{d^2y}{dx^2} + f(x)\frac{dy}{dx} + y\mathrm{f}(x) = 0 \ldots (h^{\mathrm{VII}}),$$

pone

$$y = e^{\int z dx} \ldots (h^{\mathrm{VIII}}),$$

unde

$$\frac{dy}{dx} = e^{\int z dx} . z, \; \frac{d^2y}{dx^2} = e^{\int z dx} . z^2 + e^{\int z dx} . \frac{dz}{dx};$$

immutabitur (h^{VII}) in aequationem primi ordinis

$$\frac{dz}{dx} + z^2 + zf(x) + \mathrm{f}(x) = 0 \ldots (h^{\mathrm{IX}}),$$

ex cujus integratione emerget z expressa per x, substituenda in (h^{VIII}), seu (241 ex p. 1.[a]) in

$$y = 1 + \int z dx + \frac{(\int z dx)^2}{2} + \frac{(\int z dx)^3}{2.3} + \ldots$$

Generatim data

vertetur in

$$y=y_0+a_1 z^{\alpha}+a_2 z^{\alpha+1}+a_3 z^{\alpha+2}+\ldots (h^{IV}),$$

unde

$$\frac{dy}{dx}=a_1\alpha z^{\alpha-1}+a_2(\alpha+1)z^{\alpha}+a_3(\alpha+2)z^{\alpha+1}+\ldots (h^{V});$$

in (h) substitue valores x, y, $\frac{dy}{dx}$; erit

$$(h^{V})-F(z+x_0\,,\,(h^{IV}))=0\ \ldots\ (h^{VI});$$

ex qua ordinata quoad potentias crescentes quantitatis z eruentur α, a_1, a_2, ... Ut res declaretur exemplo, proponatur aequatio

$$dy-ydx-bx^m dx=0$$

sic integranda, ut valori $x=0$ respondeat $y=0$: erunt $x_0=0$, $y_0=0$, $F(x,y)=y+bx^m$; et

$$bz^m-a_1\alpha z^{\alpha-1}+[a_1-a_2(\alpha+1)]z^{\alpha}+[a_2-a_3(\alpha+2)]z^{\alpha+1}+\ldots=0;$$

quae cum valere debeat utcumque se habet z, evanescent coefficientes singuli potentiarum z; et quia b haud evanescit per se, ideo sic erit determinanda α ut b non sit considerandus seorsum ab alio quovis coefficiente. Jam si fiat $\alpha-1=m$, vertetur aequatio in

$$[b-a_1(m+1)]z^m+[a_1-a_2(m+2)]z^{m+1}+$$
$$[a_2-a_3(m+3)]z^{m+2}+\ldots=0,$$

eruntque

$$b-a_1(m+1)=0,\ a_1-a_2(m+2)=0,$$
$$a_2-a_3(m+3)=0,\ \text{et caet.}\ \ldots$$

unde

$$a_1=\frac{b}{m+1},\ a_2=\frac{b}{(m+1)(m+2)},\ a_3=\frac{b}{(m+1)(m+2)(m+3)},\ \text{et caet...}$$

et consequenter

$$y=\frac{b}{m+1}x^{m+1}\left(1+\frac{x}{m+2}+\frac{x^2}{(m+2)(m+3)}+\ldots\right).$$

Facto $m=0$, prodibit (241 ex p. 1.^{a})

$$y=b\left(x+\frac{x^2}{2}+\frac{x^3}{2.3}+\ldots\right)=b(e^x-1).$$

195. Data aequatione secundi ordinis

$$\frac{d^2y}{dx^2}+f(x)\frac{dy}{dx}+y\mathrm{f}(x)=0\ \ldots\ (h^{\mathrm{VII}}),$$

pone

$$y=e^{\int zdx}\ \ldots\ (h^{\mathrm{VIII}}),$$

unde

$$\frac{dy}{dx}=e^{\int zdx}.z,\ \ \frac{d^2y}{dx^2}=e^{\int zdx}.z^2+e^{\int zdx}.\frac{dz}{dx};$$

immutabitur (h^{VII}) in aequationem primi ordinis

$$\frac{dz}{dx}+z^2+zf(x)+\mathrm{f}(x)=0\ \ldots\ (h^{\mathrm{IX}}),$$

ex cujus integratione emerget z expressa per x, substituenda in (h^{VIII}), seu (241 ex p. 1.^{a}) in

$$y=1+\int zdx+\frac{(\int zdx)^2}{2}+\frac{(\int zdx)^3}{2.3}+\ldots$$

Generatim data

$$F\left(x\,,\,y\,,\,\frac{dy}{dx}\,,\,\frac{d^2y}{dx^2}\,,\,\frac{d^3y}{dx^3}\,,\ldots\right)=0\ldots(h^{\mathrm{x}})\,,$$

sume

$$y=a_1x^{\alpha}+a_2x^{\beta}+a_3x^{\nu}+a_4x^{\delta}+\ldots(h^{\mathrm{xi}}):$$

hinc

$$\frac{dy}{dx}=a_1\alpha x^{\alpha-1}+a_2\beta x^{\beta-1}+a_3\nu x^{\nu-1}+\ldots(h^{\mathrm{xii}})$$

$$\frac{d^2y}{dx^2}=a_1\alpha(\alpha-1)x^{\alpha-2}+a_2\beta(\beta-1)x^{\beta-2}+\ldots(h^{\mathrm{xiii}})$$

$$\frac{d^3y}{dx^3}=a_1\alpha(\alpha-1)(\alpha-2)x^{\alpha-3}+a_2\beta(\beta-1)(\beta-2)x^{\beta-3}+\ldots(h^{\mathrm{xiv}}),$$

et caet. . . . ;

et adhibitis substitutionibus in (h^{x}),

$$F(x\,,\,(h^{\mathrm{xi}})\,,\,(h^{\mathrm{xii}})\,,\,(h^{\mathrm{xiii}})\,,\ldots)=0\ldots(\mathrm{H})\,,$$

cujus ope elaborandum ut apte determinentur α, β, . . . a_1, a_2, . . . Proponatur v. gr.

$$\frac{d^2y}{dx^2}+kyx^n=0\ldots(i):$$

erit

$$\left.\begin{array}{r}a_1\alpha(\alpha-1)x^{\alpha-2}+a_2\beta(\beta-1)x^{\beta-2}+ka_1x^{\alpha+n}+a_3\nu(\nu-1)x^{\nu-2}+\\ ka_2x^{\beta+n}+a_4\delta(\delta-1)x^{\delta-2}+ka_3x^{\nu+n}+\ldots\end{array}\right\}=0\,,$$

quae, factis

$\beta-2=\alpha+n$, $\nu-2=\beta+n$, $\delta-2=\nu+n$, et caet. . . ,

ideoque

$\beta=\alpha+n+2$, $\gamma=\alpha+2n+4$, $\delta=\alpha+3n+6$, et caet...,
vertetur in

$$\left.\begin{array}{l} a_1\alpha(\alpha-1)x^{\alpha-2}+[a_2(\alpha+n+2)(\alpha+n+1)+ka_1]x^{\alpha+n}+ \\ {[a_3(\alpha+2n+4)(\alpha+2n+3)+ka_2]x^{\alpha+2n+2}}+ \\ {[a_4(\alpha+3n+6)(\alpha+3n+5)+ka_3]x^{\alpha+3n+4}}+\dots \end{array}\right\}=0;$$

hinc

$$a_1\alpha(\alpha-1)=0\,,\ a_2(\alpha+n+2)(\alpha+n+1)+ka_1=0\,,$$
$$a_3(\alpha+2n+4)(\alpha+2n+3)+ka_2=0\,,\ \text{et caet....}$$

Ex istarum prima

$$\alpha=0\,,\ \alpha=1\,;$$

quorum valorum alter praebet

$$a_2=-\frac{ka_1}{(n+1)(n+2)}\,,\ a_3=\frac{k^2a_1}{1.2(n+1)(n+2)^2(2n+3)}\,,$$

$$a_4=-\frac{k^3a_1}{1.2.3(n+1)(n+2)^3(2n+3)(3n+5)}\,,\ \text{et caet....,}$$

alter

$$a_2=-\frac{ka_1}{(n+2)(n+3)}\,,\ a_3=\frac{k^2a_1}{1.2(n+2)^2(n+3)(2n+5)}\,,$$

$$a_4=-\frac{k^3a_1}{1.2.3(n+2)^3(n+3)(2n+5)(3n+7)}\,,\ \text{et caet...;}$$

et quia a_1 permanet omnino arbitraria, iccirco adhibita C_1 loco a_1 quoad $\alpha=0$, et C_2 quoad $\alpha=1$, provenient bina integralia incompleta, quae in summam collecta suppeditabunt integrale completum

$$\left.\begin{array}{l} y=C_1-\dfrac{kC_1}{(n+1)(n+2)}x^{n+2}+\dfrac{k^2C_1}{1.2(n+1)(n+2)^2(n+3)}x^{2n+4}-\ldots+\\ C_2x-\dfrac{kC_2}{(n+2)(n+3)}x^{n+3}+\dfrac{k^2C_2}{1.2(n+2)^2(n+3)(2n+5)}x^{2n+5}-\ldots \end{array}\right\}(i')$$

Si $n=-2$, denominatores in utraque serie fient$=0$; verum in ea qua sumus hypothesi erunt

$$\alpha=\beta=\nu=\delta=\ldots,\ y=(a_1+a_2+a_3+\ldots)x^{\alpha}=Ax^{\alpha},$$

$$a_1\alpha(\alpha-1)x^{\alpha-2}+(a_2\alpha(\alpha-1)+ka_1)x^{\alpha-2}+(a_3\alpha(\alpha-1)+$$

$$ka_2)x^{\alpha-2}+\ldots=A(\alpha(\alpha-1)+k)x^{\alpha-2}=0,\ \alpha(\alpha-1)+k=0,$$

$$\alpha=\frac{1+\sqrt{[1-4k]}}{2}=\alpha',\ \alpha=\frac{1-\sqrt{[1-4k]}}{2}=\alpha'':$$

et quia A permanet arbitraria, ideo

$$y=C_1x^{\alpha'}+C_2x^{\alpha''}.$$

196. Notentur haec tria : 1.° e duobus integralibus incompletis aequationis (h^{VII}) dato altero

$$y=C_1\chi(x)\ldots(i''),$$

determinatur alterum. Exhibeatur quaesitum integrale per

$$y=\chi(x)\varphi(x)\ldots(i''');$$

denotat $\varphi(x)$ functionem incognitam: substitutis valoribus

$$y,\ \frac{dy}{dx},\ \frac{d^2y}{dx^2}$$

in (h^{VII}) ex (i'''), proveniet

$$\varphi(x)[\chi''(x)+\chi'(x)f(x)+\chi(x)\mathfrak{f}(x)]+$$

$$2\chi'(x)\varphi'(x)+\chi(x)\varphi''(x)+f(x)\chi(x)\varphi'(x)=0:$$

substitutis autem iisdem valoribus ex (i''), habemus

$$\chi''(x)+\chi'(x)f(x)+\chi(x)\mathfrak{f}(x)=0\ ;$$

ergo

$$2\chi'(x)\varphi'(x)+\chi(x)\varphi''(x)+f(x)\chi(x)\varphi'(x)=0,$$

unde

$$\frac{2\chi'(x)}{\chi(x)}+\frac{\varphi''(x)}{\varphi'(x)}+f(x)=0\ ;$$

quae scribi potest in hunc modum

$$\frac{2d\chi(x)}{\chi(x)}+\frac{d\varphi'(x)}{\varphi'(x)}=-f(x)dx\ ;$$

sumptisque integralibus,

$$2\mathrm{L}(\chi(x))+\mathrm{L}(\varphi'(x))+\mathrm{L}(\mathrm{C}_2)=-\int f(x)dx,$$

ex qua

$$\varphi'(x)=\frac{e^{-\int f(x)dx}}{\mathrm{C}_2\chi^2(x)},\text{ seu } d\varphi(x)=\frac{e^{-\int f(x)dx}}{\mathrm{C}_2\chi^2(x)}dx\ ;$$

et sumptis iterum integralibus,

$$\varphi(x)=\int\frac{e^{-\int f(x)dx}}{\mathrm{C}_2\chi^2(x)}dx.$$

Propterea

$$y=\chi(x)\int\frac{e^{-\int f(x)dx}}{\mathrm{C}_2\chi^2(x)}dx\ldots(i^{\mathrm{IV}}),$$

integrale quaesitum.

2.° si (i'') et (i^{IV}) colligantur in summam, scribaturque C_2 loco $\frac{1}{C_2}$, emerget

$$y=\chi(x)\left(C_1+C_2\int\frac{e^{-\int f(x)dx}}{\chi^2(x)}dx\right)\ldots(i^{\text{V}}),$$

integrale completum aequationis (h^{VII}). Detur v. gr.

$$\frac{d^2y}{dx^2}-\frac{1}{x(L(x)-1)}\frac{dy}{dx}+\frac{y}{x^2(L(x)-1)}=0$$

una cum altero ex ejus integralibus incompletis

$$y=C_1x:$$

erunt

$$\chi(x)=x\,,\ f(x)=\frac{1}{x(L(x)-1)}\,,\ (i^{\text{IV}})=x\int\frac{e^{\int\frac{dx}{x(L(x)-1)}}}{C_2x^2}dx=$$

$$x\int\frac{e^{\int\frac{dL(x)}{L(x)-1}}}{C_2x^2}dx=x\int\frac{e^{L(L(x)-1)}}{C_2x^2}dx=x\int\frac{L(x)-1}{C_2x^2}dx;$$

ideoque (146)

$$y=-\frac{L(x)}{C_2},$$

alterum integrale incompletum; et

$$y=C_1x-C_2L(x),$$

integrale completum.

3.° quoniam in (h^{VII}) factis $f(x)=0$, $f(x)=kx^n$, immutatur (h^{VII}) in (i), iccirco si (i'') denotat alterum e duobus integralibus incompletis aequationis (i), ad habendum alterum satis erit ponere $f(x)=0$ in (i^{IV});

eadem vero positio in (i^v) dabit integrale completum ipsius (i).

Quibus annotatis, sit in (i) vel $n=-1$, aut $n=-\frac{3}{2}$, vel et caet...; evanescent denominatores in prima tantum serie (i'); restabitque secunda, alterum videlicet e duobus integralibus incompletis aequationis (i): inde autem habebitur et alterum; quae integralia in summam collecta suppeditabunt integrale completum. Quodsi vel $n=-3$, aut $n=-\frac{5}{2}$, vel et caet...; evanescent denominatores in secunda tantum serie, restabitque prima, alterum videlicet et caet...

DE SOLUTIONIBUS PARTICULARIBUS AEQUATIONUM DIFFERENTIALIUM CUJUSCUMQUE ORDINIS.

197. Data

$$f(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots \frac{d^ny}{dx^n}) = 0 \ldots (k),$$

cujus integrale

$$F(x, y, C_1, C_2, \ldots C_n) = 0 \ldots (k'),$$

ubi ab

$$F=0, \frac{1}{dx}dF=0, \frac{1}{dx^2}d^2F=0, \ldots \frac{1}{dx^n}d^nF=0 \ldots (k'')$$

eliminentur $C_1, C_2, \ldots C_n$, inde profluet (k). Sic v. gr. data

$$\frac{d^2y}{dx^2} + ay - 1 = 0,$$

erit (187. II.°)

$$F = ay - 1 + \cos x\sqrt{a} - (C_1 \sin x\sqrt{a} + C_2 \cos x\sqrt{a})\sqrt{a} = 0,$$

et consequenter

$$\frac{1}{dx}dF = a\frac{dy}{dx} - \sqrt{a}\sin x\sqrt{a} - (C_1 \cos x\sqrt{a} - C_2 \sin x\sqrt{a})a = 0,$$

$$\frac{1}{dx^2}d^2F = \frac{d^2y}{dx^2} - \cos x\sqrt{a} + (C_1 \sin x\sqrt{a} + C_2 \cos x\sqrt{a})\sqrt{a} = 0;$$

quarum prima et tertia manifeste praebent

$$\frac{d^2y}{dx^2} + ay - 1 = 0.$$

Hoc posito, differentientur aequationes (k''), exempta ultima, quoad C_1, C_2, ... C_n, et quae resultant differentialia denotentur per

$$d_c F, \; d_c \frac{1}{dx}dF, \; \ldots \; d_c \frac{1}{dx^{n-1}}d^{n-1}F:$$

ad eliminandas C_1, C_2, ... C_n perinde erit sive adhibeantur (k''), sive

$$F = 0, \; \frac{1}{dx}dF + d_c F = 0, \; \frac{1}{dx^2}d^2F + d_c \frac{1}{dx}dF = 0, \ldots$$

$$\frac{1}{dx^n}d^nF + d_c \frac{1}{dx^{n-1}}d^{n-1}F = 0,$$

modo tamen exsistant

$$d_c F = 0, \; d_c \frac{1}{dx}dF = 0, \; \ldots \; d_c \frac{1}{dx^{n-1}}d^{n-1}F = 0,$$

seu

$$\frac{1}{dC_n}d_c F = 0, \; \frac{1}{dC_n}d_c \frac{1}{dx}dF = 0, \ldots \frac{1}{dC_n}d_c \frac{1}{dx^{n-1}}d^{n-1}F = 0 \ldots (k''').$$

Inde colligitur, si ab (k''') eliminantur quantitates

$$\frac{dC_1}{dC_n},\ \frac{dC_2}{dC_n},\ \ldots\ \frac{dC_{n-1}}{dC_n}$$

ut exsurgat differentialis aequatio ordinis $\overline{n-1}^{simi}$

$$f(x, y, C_1, C_2, \ldots C_n, \frac{dy}{dx}, \ldots \frac{d^{n-1}y}{dx^{n-1}})=0 \ldots (k^{IV}),$$

itemque ab

$$f=0,\ F=0,\ \frac{1}{dx}dF=0,\ \ldots\ \frac{1}{dx^{n-1}}d^{n-1}F=0 \ldots (k^{V})$$

eliminantur C_1, C_2, ... C_n ut prodeat differentialis aequatio ordinis $\overline{n-1}^{simi}$

$$\chi(x, y, \frac{dy}{dx}, \ldots \frac{d^{n-1}y}{dx^{n-1}})=0 \ldots (k^{VI}),$$

colligitur inquam fore (k^{VI}) particularem solutionem aequationis (k). Proponatur v. gr.

$$F=y-\frac{C_1}{2}x^2-C_2x-C_1^2-C_2^2=0,$$

ex qua simul et ex duabus

$$\frac{1}{dx}dF=0,\ \frac{1}{dx^2}d^2F=0$$

emergit (197)

$$f=y-x\frac{dy}{dx}+\frac{x^2}{2}\frac{d^2y}{dx^2}-(\frac{d^2y}{dx^2})^2-(\frac{dy}{dx}-x\frac{d^2y}{dx^2})^2=0 \ldots (a).$$

Habemus

$$\frac{1}{dC_2}d_cF=-(\frac{x^2}{2}+2C_1)\frac{dC_1}{dC_2}-x-2C_2=0,$$

$$\frac{1}{dC_2}d_o\frac{1}{dx}dF = -x\frac{dC_1}{dC_2} - 1 = 0 \,;$$

unde

$$f = \left(\frac{x^2}{2} + 2C_1\right)\frac{1}{x} - x - 2C_2 = 0 \,:$$

et eliminatis C_1, C_2 ab

$$f = 0 \,,\; F = 0 \,,\; \frac{1}{dx}dF = 0 \,,$$

prodibit

$$\chi = y(x^2+1) + \frac{x}{16} - \left(\frac{dy}{dx}\right)^2 - \left(x + \frac{x^3}{2}\right)\frac{dy}{dx} = 0$$

particularis solutio aequationis (a).

Attendenti patebit praeter (k^{VI}) etiam ejus integralia variorum ordinum

$$\left.\begin{array}{l} \chi_1\left(x\,,\; y\,,\; C_1\,,\; \dfrac{dy}{dx}\,,\ldots\, \dfrac{d^{n-2}y}{dx^{n-2}}\right) = 0\,, \\[2ex] \chi_2\left(x\,, y\,, C_1\,, C_2\,, \dfrac{dy}{dx}\,,\ldots\, \dfrac{d^{n-3}y}{dx^{n-3}}\right) = 0\,, \\[2ex] \text{et caet.} \ldots\ldots \\[1ex] \chi_{n-1}(x\,,\; y\,,\; C_1\,,\; C_2\,,\;\ldots C_{n-1}) = 0 \end{array}\right\} (k^{VII})$$

fore solutiones particulares aequationis (k) : unde infertur amplissimam omnium particularium solutionum ipsius (k) haud plures continere posse constantes arbitrarias quam $n-1$.

Evenit aliquando ut in (k^{IV}) nulla ex arbitrariis $C_1, \ldots C_n$ reperiatur; tunc χ recidet in f. Sit v. gr.

$$f = yx^2\frac{d^2y}{dx^2} - 2x^2\frac{dy^2}{dx^2} + 6xy\frac{dy}{dx} - 6y^2 = 0\,,$$

cujus integrale (182. 4.°)

$$F = xy + C_1 x^3 + C_2 y = 0:$$

erunt

$$\frac{1}{dC_2} d_c F = x^3 \frac{dC_1}{dC_2} + y = 0, \frac{1}{dC_2} d_c \frac{1}{dx} dF = 3x^2 \frac{dC_1}{dC_2} + \frac{dy}{dx} = 0;$$

hinc particularis solutio

$$f = \frac{1}{3}\frac{dy}{dx} - \frac{y}{x} = 0, \text{ seu } \frac{1}{3}\frac{dy}{y} - \frac{dx}{x} = 0;$$

quae integrata suppeditat solutionem alteram particularem

$$y = C_2 x^3.$$

Si in (k^{IV}) essent tantummodo $n-m$ arbitrariae $C_1, C_2, \ldots C_{n-m}$, transitus ab (k^{IV}) ad (k^{VI}) fieret per solas

$$f = 0, \frac{1}{dx^{n-1}} d^{n-1}F = 0, \frac{1}{dx^{n-2}} d^{n-2}F = 0, \ldots \frac{1}{dx^m} d^m F = 0.$$

198. Ponatur aequatio (k) expleri, sive loco y adhibeatur

$$F_1(x, C_1, C_2, \ldots C_m),$$

sive

$$F_1(x, C_1, C_2, \ldots C_m) + \sigma\varphi(x, C_1, C_2, \ldots C_m);$$

denotat m numerum haud $> n$, σ quantitatem infinitesimam. Expressis compendii causa

$$\frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots \frac{d^ny}{dx^n}$$

per $y, y', y'', \ldots y^{(n)}$ ut (k) scribi possit in hunc modum

$$f(x, y, y', y'', \ldots y^{(n)}) = 0 \ldots (k_1),$$

certe in ea qua sumus hypothesi subsistet (k_1) etiam quum loco y, y'; y'', ... $y^{(n)}$ substituuntur

$$y+\sigma\varphi\,,\, y'+\sigma\varphi'\,,\, y''+\sigma\varphi''\,,\,\ldots\, y^{(n)}+\sigma\varphi^{(n)}:$$

designato igitur per μ primo membro aequationis (k_1), erit (63)

$$\mu+\sigma\left(\frac{d\mu}{dy}\varphi+\frac{d\mu}{dy'}\varphi'+\ldots+\frac{d\mu}{dy^{(n)}}\varphi^{(n)}\right)+\frac{\sigma^2}{2}\left(\frac{d^2\mu}{dy^2}\varphi^2(x)+\right.$$

$$\left.\frac{d^2\mu}{dy'^2}\varphi'^2(x)+\ldots+2\frac{d^2\mu}{dydy'}\varphi(x)\varphi'(x)+\ldots+\sigma_1\right)=0\;;$$

denotat σ_1 infinitesimam quantitatem. Habetur autem $\mu=0$; hinc

$$\frac{d\mu}{dy}\varphi+\frac{d\mu}{dy'}\varphi'+\ldots+\frac{d\mu}{dy^{(n)}}\varphi^{(n)}+\frac{\sigma}{2}\left(\frac{d^2\mu}{dy^2}\varphi^2(x)+\right.$$

$$\left.\frac{d^2\mu}{dy'^2}\varphi'^2(x)+\ldots+2\frac{d^2\mu}{dydy'}\varphi(x)\varphi'(x)+\ldots+\sigma_1\right)=0\;;$$

et facta $\sigma=0$,

$$\frac{d\mu}{dy^{(n)}}\varphi^{(n)}+\frac{d\mu}{dy^{(n-1)}}\varphi^{(n-1)}+\ldots+\frac{d\mu}{dy'}\varphi'+\frac{d\mu}{dy}\varphi=0\ldots(k^{\text{VIII}}).$$

Jamvero coefficiens $\frac{d\mu}{dy^{(n)}}$ functionis derivatae $\varphi^{(n)}$ vel permanet aliquis, vel evanescit: in primo casu, (k^{VIII}) utpote spectans ad *nsimum* ordinem praebebit φ complectentem n arbitrarias; eritque $m=n$, et consequenter (197) $y=\text{F}_1$ haud exsistet particularis solutio, sed integrale completum: in secundo, (k^{VIII}) ad *nsimum* non pertingens ordinem suppeditabit φ complectentem minus quam n arbitrarias; erit idcirco $m<n$, et $y=\text{F}_1$ particularis solutio.

199. Inferimus, quotiescumque (k_1) admittit solutiones particulares, fore

$$\frac{d\mu}{dy^{(n)}}=0\ldots(k^{\text{IX}}).$$

Ad haec: quia (45)

$$\frac{1}{dx}d\mu=\frac{d\mu}{dx}+\frac{d\mu}{dy}y'+\frac{d\mu}{dy'}y''+\ldots$$

$$+\frac{d\mu}{dy^{(n-1)}}y^{(n)}+\frac{d\mu}{dy^{(n)}}y^{(n+1)}=0,$$

ideo in eadem solutionum particularium hypothesi erit ob (k^{IX})

$$\frac{d\mu}{dx}+\frac{d\mu}{dy}y'+\ldots+\frac{d\mu}{dy^{(n-1)}}y^{(n)}=0;$$

ac proinde

$$y^{(n+1)}=\frac{d^{n+1}y}{dx^{n+1}}=\frac{0}{0}\ldots(k^{\text{X}}).$$

Ope (k^{IX}), vel (k^{X}) eliciuntur solutiones particulares ab ipsa aequatione differentiali quin cognoscatur ejus integrale.

Exempla.

I.° Sit

$$\frac{d^2y}{dx^2}+2x-2\sqrt{[x^2+\frac{dy}{dx}]}=0 \text{ seu } y''+2x-2\sqrt{[x^2+y']}=0,$$

quae transformetur in

$$\mu=y''\sqrt{[x^2+y']}+2x\sqrt{[x^2+y']}-2(x^2+y')=0;$$

inde (k^{IX})

$$\frac{d\mu}{dy''}=\sqrt{[x^2+y']}=0,$$

et consequenter

$$dy=-x^2dx\ ,\ y=C_1-\frac{x^3}{3}:$$

haec, cum aequationi datae satisfaciat, erit ejus solutio particularis.

II.° Sit aequatio

$$\mu=yy'-y'\sqrt{[x^2+y^2-a^2]}+x=0\,,$$

jam pertractata (169); erit (k^{IX})

$$\frac{d\mu}{dy'}=y-\sqrt{[x^2+y^2-a^2]}=0\,,$$

unde

$$x^2-a^2=0.$$

Quod si aequatio transformetur in

$$\mu=yy'\sqrt{[x^2+y^2-a^2]}-y'(x^2+y^2-a^2)+x\sqrt{[x^2+y^2-a^2]}=0,$$

exsistet

$$\frac{d\mu}{dy'}=(y-\sqrt{[x^2+y^2-a^2]})\sqrt{[x^2+y^2-a^2]}=0\,,$$

ex qua praeter $x^2-a^2=0$, emergit etiam

$$x^2+y^2-a^2=0.$$

III.° Sit

$$xy''^2-2y'y''+x=0\,,$$

ex qua

$$y'''=\frac{y''^2-1}{2(xy''-y')}\,;$$

erunt (k^{X})

$$y''^2-1=0\,,\ xy''-y'=0:$$

et eliminata y'',

$$y'^2-x^2=0\ ,\ dy=xdx\ ,\ y=C_1+\frac{x^2}{2}\ .$$

IV.° Proponatur demum

$$y=xy'-a\sqrt{[1+y'^2]}=0\ :$$

erit

$$y''=\frac{0}{x\sqrt{[1+y'^2]}-ay'},\ \text{ideoque } (k^x)\ x\sqrt{[1+y'^2]}-ay'=0\ ;$$

unde

$$y'^2=\frac{x^2}{a^2-x^2}\ ,\ y'=\pm\frac{x}{\sqrt{[a^2-x^2]}}\ ;$$

quarum ope eliminata y' ab aequatione proposita, obvenient solutiones particulares

$$y^2(a^2-x^2)-(x^2+a^2)^2=0\ ,\ y^2+x^2-a^2=0.$$

DE INTEGRATIONE QUARUMDAM DIFFERENTIALIUM PARTIALIUMQUE AEQUATIONUM; DEQUE HUJUSMODI AEQUATIONUM INTEGRATIONE PER SERIES; UBI ET NONNULLORUM INTEGRALIUM DEFINITORUM VALORES DETERMINANTUR.

200. Habita z pro functione binarum x, y ut sit (45 : 51)

$$dz=\frac{d_xz}{dx}dx+\frac{d_yz}{dy}dy\ ,$$

et consequenter

$$\left.\begin{array}{r}\dfrac{d_xz}{dx}+\dfrac{d_yz}{dy}\dfrac{dy}{dx}=\dfrac{dz}{dx}\,,\ \dfrac{d_xz}{dx}\dfrac{dx}{dy}+\dfrac{d_yz}{dy}=\dfrac{dz}{dy}\,,\\ \dfrac{d_xz}{dx}\dfrac{dx}{dz}+\dfrac{d_yz}{dy}\dfrac{dy}{dz}=1\,,\end{array}\right\}(a')$$

proponitur integranda aequatio

$$P\frac{d_xz}{dx}+Q\frac{d_yz}{dy}=R\ \ldots\ (a'')\,,$$

in qua P, Q, R denotant functiones variabilium x, y et z.

Exhibeatur integrale aequationis propositae per

$$\chi(x\,,y\,,z)=0\ \ldots\ (a''')\,:$$

conferentes (a'') cum prima (a') habemus

$$\frac{Q}{P}=\frac{dy}{dx}\,,\ \frac{R}{P}=\frac{dz}{dx}\,,$$

ideoque

$$Pdy-Qdx=0\,,\ Pdz-Rdx=0\,;$$

quarum integralibus (153 ...) designatis per μ_1, μ_2, erunt

$$\mu_1=C_1\,,\ \mu_2=C_2\,:$$

hinc

$$x=f_1(z\,,C_1\,,C_2)\,,\ y=f_2(z\,,C_1\,,C_2)\,;$$

et adhibitis substitutionibus in (a'''),

$$\chi(f_1\,,f_2\,,z)=0\ \ldots\ (a^{IV}).$$

Jamvero nisi z per se dispareat ab (a^{IV}), necesse est ut z exsistat constans; quod cum nequeat admitti, disparebit igitur z per se ab (a^{IV}), ipsaque (a^{IV}) vertetur in

$$\chi_1(C_1\,,C_2)=0\ \text{seu}\ \chi_1(\mu_1\,,\mu_2)=0\,;$$

ac proinde integrale quaesitum (54)

$$\mu_1=\varphi(\mu_2)\,,\ \text{vel etiam}\ \mu_2=\varphi(\mu_1).$$

Si (a'') compararetur vel cum secunda, vel cum tertia (a'), prodirent in 1.° casu

$$Qdz-Rdy=0\,,\ Pdy-Qdx=0\,,$$

in 2.°

$$Pdz-Rdx=0\,,\ Qdz-Rdy=0.$$

Functiones arbitrariae determinantur ex assignatis conditionibus: sit v. gr. ita determinanda $\varphi(\mu_2)$ in integrali, $\mu_1=\varphi(\mu_2)$, ut, posita $F(x, y, z)=0$, habeatur simul $f(x, y, z)=0$. Pone $\mu_2=\nu$, atque ex

$$\mu_2=\nu,\ F(x, y, z)=0\,,\ f(x, y, z)=0$$

erue x, y, z expressas per ν; et adhibitis substitutionibus in μ_1, vertetur μ_1 in functionem V unius ν; eritque $\varphi(\nu)=V$: tum in hac aequatione restitue valorem ν datum per x, y et z; prodibit φ determinata. Istiusmodi determinatio in id manifeste recidit, ut superficies

$$\mu_1=\varphi(\mu_2)$$

adstringatur transire per datam lineam

$$\{F(x, y, z)=0\,,\ f(x, y, z)=0.\}$$

Quo plures sunt arbitrariae functiones determinandae, eo etiam plures requiruntur conditiones.

Exempla.

I.° Detur

$$x\frac{d_xz}{dx}+y\frac{d_yz}{dz}=az:$$

erunt $P=x$, $Q=y$, $R=az$, ideoque $xdy-ydx=0$, $xdz-azdx=0$, seu $\frac{dy}{y}-\frac{dx}{x}=0$, $\frac{dz}{z}-a\frac{dx}{x}=0$; istarum prima suppeditat $L(\frac{y}{x})=L(C_1)$, secunda $L(\frac{z}{x^a})=L(C_2)$, nimirum $\mu_1=\frac{y}{x}$, $\mu_2=\frac{z}{x^a}$; et consequenter $z=x^a\varphi(\frac{y}{x})$.

II.°

$$xy\frac{d_yz}{dy}-x^2\frac{d_xz}{dx}=y^2:$$

erunt $P=-x^2$, $Q=xy$, $R=y^2$, propterea $x^2dy+xydx=0$, $x^2dz+y^2dx=0$. Harum prima vertitur in $\frac{dy}{y}+\frac{dx}{x}=0$, ex qua $xy=C_1=\mu_1$: substituto valore $y=\frac{C_1}{x}$ in secunda, proveniet

$$x^2dz+\frac{C_1^2}{x^2}dx=0 \text{ seu } dz+\frac{C_1^2}{x^4}dx=0,$$

ex qua

$$z-\frac{C_1^2}{3x^3}=C_2 \text{ seu } z-\frac{y^2}{3x}=C_2=\mu_2;$$

hinc

$$z-\frac{y^2}{3x}=\varphi(xy).$$

III.°

$$y^2\frac{d_xz}{dx}=yz:$$

erunt $P=y^2$, $Q=0$, $R=yz$; propterea $dy=0$, $ydz-zdx=0$. Istarum prima suppeditat $y=C_1=\mu_1$, ideoque secunda immutabitur in $C_1\frac{dz}{z}-dx=0$; unde $yL(z)-x=C_2=\mu_2$, et consequenter $yL(z)-x=\varphi(y)$.

201. Habita z pro functione trium variabilium x, y, t ut sit

$$dz=\frac{d_xz}{dx}dx+\frac{d_yz}{dy}dy+\frac{d_tz}{dt}dt,$$

ac proinde

$$\left.\begin{array}{l}\dfrac{d_xz}{dx}+\dfrac{d_yz}{dy}\dfrac{dy}{dx}+\dfrac{d_tz}{dt}\dfrac{dt}{dx}=\dfrac{dz}{dx}\,,\\[2ex] \dfrac{d_xz}{dx}\dfrac{dx}{dy}+\dfrac{d_yz}{dy}+\dfrac{d_tz}{dt}\dfrac{dt}{dy}=\dfrac{dz}{dy}\,,\\[2ex] \text{et caet.}\ldots,\end{array}\right\}(a^{\mathrm{v}})$$

proponitur integranda aequatio

$$N\frac{d_xz}{dx}+P\frac{d_yz}{dy}+Q\frac{d_tz}{dt}=R\ldots(a^{\mathrm{vi}}),$$

in qua N, P, Q, R denotant functiones variabilium x, y, z, t.

Designetur ejus integrale per

$$\chi(x,y,z,t)=0\ldots(a^{\mathrm{vii}}):$$

comparantes (a^{vi}) cum prima (a^{v}) habemus

$$\frac{P}{N}=\frac{dy}{dx},\ \frac{Q}{N}=\frac{dt}{dx},\ \frac{R}{N}=\frac{dz}{dx},$$

ideoque

$$Ndy-Pdx=0,\ Ndt-Qdx=0,\ Ndz-Rdx=0;$$

quarum integralibus (153 ...) designatis per μ_1, μ_2, μ_3, erunt

$$\mu_1=C_1\ ,\ \mu_2=C_2\ ,\ \mu_3=C_3\ ;$$

hinc

$$x=f_1(z,C_1,C_2,C_3), y=f_2(z,C_1,C_2,C_3), t=f_3(z,C_1,C_2,C_3);$$

et adhibitis substitutionibus in (a^{VII}),

$$\chi(f_1\ ,\ f_2\ ,\ f_3\ ,\ z)=0\ \ldots\ (a^{VIII})\ .$$

Jamvero nisi z dispareat per se ab (a^{VIII}) necesse est z exsistat constans; quod cum nequeat admitti, disparebit igitur z per se ab (a^{VIII}), ipsaque (a^{VIII}) vertetur in

$$\chi_1\,(C_1\ ,\ C_2\ ,\ C_3)=0 \text{ seu } \chi_1(\mu_1\ ,\ \mu_2\ ,\ \mu_3)=0\ :$$

unde quaesitum integrale

$$\mu_1=\varphi(\mu_2,\mu_3), \text{ vel } \mu_2=\varphi(\mu_1,\mu_3), \text{ vel etiam } \mu_3=\varphi(\mu_1,\mu_2).$$

Si (a^{VI}) compararetur cum secunda v. gr. (a^{V}), prodirent

$$Ndy-Pdx=0\ ,\ Pdt-Qdy=0\ ,\ Pdz-Rdy=0.$$

Ut res declaretur exemplo, sit

$$x\frac{d_xz}{dx}+(z+t)\frac{d_yz}{dy}+(z+y)\frac{dz}{dt}=y+t:$$

erunt $N=x$, $P=z+t$, $Q=z+y$, $R=y+t$; proinde

$$xdy-(z+t)dx=0\ ,\ xdt-(z+y)dx=0,\ xdz-(y+t)dx=0\ ,$$

ex quibus cum aperte habeamus

$$x(dy-dt)+(y-t)dx=0\ ,\ x(dy-dz)+(y-z)dx=0\ ,$$

$$x(dy+dt+dz)-2(y+t+z)dx=0\ ,$$

idcirco

$$x(y-t)=\mu_1\ ,\ x(y-z)=\mu_2\ ,\ \frac{y+t+z}{x^2}=\mu_3\ ,$$

et

$$y+t+z=x^z\varphi(xy-xt\,,\;xy-xz).$$

202. Habita denuo z pro functione duarum x, y sit integranda

$$F\left(x\,,\;y\,,\;z\,,\;\frac{d_x z}{dx}\,,\;\frac{d_y z}{dy}\right)=0\;\ldots\;(a^{IX}).$$

Expressis $\frac{d_x z}{dx}$, $\frac{d_y z}{dy}$ compendii causa per z'_x, z'_y, et resoluta (a^{IX}) quoad z'_y, exsistat

$$z'_y=F_1(x\,,\;y\,,\;z\,,\;z'_x)\;\ldots\;(a^{X});$$

unde

$$\left.\begin{array}{l} dz'_y=Hdx+Kdy+Mdz+Ndz'_x= \\ Hdx+Kdy+Mdz+N(d_x z'_x+d_y z'_x+d_z z'_x). \end{array}\right\}(a^{XI})$$

Aequatio (200)

$$dz-z'_x dx-z'_y dy=0\;\ldots\;(a^{XII})$$

suppeditat (170. θ_{1})

$$\frac{d_x z'_y}{dx}-\frac{d_y z'_x}{dy}-z'_y\frac{d_z z'_x}{dz}+z'_x\frac{d_z z'_y}{dz}=0\;\ldots\;(a^{XIII});$$

(a^{XI}) praebet

$$\frac{d_x z'_y}{dx}=H+N\frac{d_x z'_x}{dx}\,,\;\frac{d_z z'_y}{dz}=M+N\frac{d_z z'_x}{dz}\,;$$

et in (a^{XIII}) substitutis valoribus $\frac{d_x z'_y}{dx}$, $\frac{d_z z'_y}{dz}$, necnon z'_y ex (a^{X}), obveniet

$$N\frac{d_x z'_x}{dx}-\frac{d_y z'_x}{dy}+\frac{d_z z'_x}{dz}(Nz'_x-F_1)+Mz'_x+H=0\;\ldots\;(a^{XIV}),$$

quae dabit (201) z'_x expressam per x, y, z; inde

autem ob (a^{x}) habebitur et z'_y expressa per x, y, z. Jam si valores z'_x, z'_y substituuntur in (a^{xii}) ipsa (a^{xii}) fiet aequatio differentialis ordinaria, cujus integrale (172) erit integrale quaesitum.

203. Proponatur nunc integranda aequatio secundi ordinis

$$\frac{d^2{}_x z}{dx^2}+a\frac{d_x d_y z}{dx dy}+b\frac{d^2{}_y z}{dy^2}=f(x,y) \ldots (a^{xv}),$$

in qua a, b sunt constantes.

Sume indeterminatam α, et pone

$$\frac{d_x z}{dx}+\alpha\frac{d_y z}{dy}=\mathrm{f} \ldots (a^{xvi});$$

habebis

$$\frac{d^2{}_x z}{dx^2}+\alpha\frac{d_x d_y z}{dx dy}=\frac{d_x \mathrm{f}}{dx},\ \frac{d_x d_y z}{dx dy}+\alpha\frac{d^2{}_y z}{dy^2}=\frac{d_y \mathrm{f}}{dy},$$

ac proinde

$$\frac{d^2{}_x z}{dx^2}=\frac{d_x \mathrm{f}}{dx}-\alpha\frac{d_x d_y z}{dx dy},\ b\frac{d^2{}_y z}{dy^2}=\frac{b}{\alpha}\,\frac{d_y \mathrm{f}}{dy}-\frac{b}{\alpha}\,\frac{d_x d_y z}{dx dy}:$$

valores istos $\frac{d^2{}_x z}{dx^2}$, $b\frac{d^2{}_y z}{dy^2}$ substitue in (a^{xv}); proveniet

$$\frac{d_x \mathrm{f}}{dx}+\frac{b}{\alpha}\,\frac{d_y \mathrm{f}}{dy}+\left(a-\alpha-\frac{b}{\alpha}\right)\frac{d_x d_y z}{dx dy}=f(x,y),$$

quae, facto

$$a-\alpha-\frac{b}{\alpha}=0 \ldots (a^{xvii}),$$

vertetur in

$$\frac{d_x \mathrm{f}}{dx}+\frac{b}{\alpha}\,\frac{d_y \mathrm{f}}{dy}=f(x,y) \ldots (a^{xviii}).$$

Jam si α_1, α_2 denotant radices aequationis (a^{xvii}), erit (146 ex p. 1.^{a}) $\alpha_1\alpha_2=b$; et in (a^{xvi}), (a^{xviii}) substituta v. gr. α_1 loco α, exsistent

$$\frac{d_x z}{dx}+\alpha_1\frac{d_y z}{dy}=\mathrm{f}\,,\quad \frac{d_x \mathrm{f}}{dx}+\alpha_2\frac{d_y \mathrm{f}}{dy}=f(x\,,y)\ \ldots\ (a^{xix})\,;$$

hinc (200) quoad primam (a^{xix})

$$dy-\alpha_1 dx=0\,,\quad dz-\mathrm{f}dx=0\ \ldots\ (a^{xx})\,,$$

quoad secundam

$$dy-\alpha_2 dx=0\,,\quad d\mathrm{f}-f(x\,,y)dx=0\ \ldots\ (a^{xxi}).$$

E binis (a^{xxi}) profluunt

$$y-\alpha_2 x=\mathrm{C}_1\,,\quad \mathrm{f}-\int f(x\,,\ \mathrm{C}_1+\alpha_2 x)dx=\mathrm{C}'\,;$$

sit

$$\int f(x\,,\mathrm{C}_1+\alpha_2 x)dx=\chi(x\,,\mathrm{C}_1)\ \ldots\ (a^{xxii})\,;$$

erit (200)

$$\mathrm{f}-\chi(x\,,\ \mathrm{C}_1)=\varphi(y-\alpha_2 x)\,,$$

scilicet

$$\mathrm{f}=\chi(x\,,y-\alpha_2 x)+\varphi(y-\alpha_2 x)\ \ldots\ (a^{xxiii}).$$

E binis (a^{xx}) prodeunt

$$y-\alpha_1 x=\mathrm{C}_2\,,\quad z-\int \mathrm{f}dx=\mathrm{C}''\,;$$

unde (200)

$$z-\int \mathrm{f}dx=\varphi_1(y-\alpha_1 x)\,,$$

et consequenter ob (a^{xxiii})

$$z=\int\chi(x\,,\mathrm{C}_2+\alpha_1 x-\alpha_2 x)dx+$$
$$\int\varphi(\mathrm{C}_2+\alpha_1 x-\alpha_2 x)dx+\varphi_1(y-\alpha_1 x)\,.$$

Est autem

$$\int\varphi(\mathrm{C}_2+\alpha_1 x-\alpha_2 x)dx=\psi(\mathrm{C}_2+\alpha_1 x-\alpha_2 x)=\psi(y-\alpha_2 x)\,;$$

adhibita igitur arbitraria φ loco arbitrariae ψ, quaesitum integrale sic exprimetur

$$z = \int \chi(x, C_2 + \alpha_1 x - \alpha_2 x)dx + \varphi(y - \alpha_2 x) + \varphi_1(y - \alpha_1 x).$$

204. Nonnulla subjungimus circa aequationem

$$\frac{d_x d_y z}{dx\,dy} + P\frac{d_x z}{dx} + Q\frac{d_y z}{dy} + Rz + S = 0 \ldots (a_1),$$

in qua P, Q, R, S denotant functiones duarum x, y. Ac 1.° sume

$$z = e^v u, \text{ unde } \frac{d_x z}{dx} = e^v\left(u\frac{d_x v}{dx} + \frac{d_x u}{dx}\right), \frac{d_y z}{dy} =$$

$$e^v\left(u\frac{d_y v}{dy} + \frac{d_y u}{dy}\right), \frac{d_x d_y z}{dx\,dy} = e^v\frac{d_y v}{dy}\left(u\frac{d_x v}{dx} + \frac{d_x u}{dx}\right) +$$

$$e^v\left(\frac{d_y u}{dy}\frac{d_x v}{dx} + u\frac{d_x d_y v}{dx\,dy} + \frac{d_x d_y u}{dx\,dy}\right);$$

transformabitur (a_1) in

$$\left.\begin{array}{l} e^v\left(u\dfrac{d_y v}{dy} + \dfrac{d_y u}{dy}\right)\left(\dfrac{d_x v}{dx} + Q\right) + e^v\left[\dfrac{d_x d_y u}{dx\,dy} + \right. \\ \left.\dfrac{d_x u}{dx}\left(\dfrac{d_y v}{dy} + P\right)\right] + ue^v\left(\dfrac{d_x d_y v}{dx\,dy} + P\dfrac{d_x v}{dx} + R\right) + S, \end{array}\right\} = 0 \ldots (a_2)$$

quae potest etiam scribi in hunc modum

$$\left.\begin{array}{l} e^v\left(u\dfrac{d_x v}{dx} + \dfrac{d_x u}{dx}\right)\left(\dfrac{d_y v}{dy} + P\right) + e^v\left[\dfrac{d_x d_y v}{dx\,dy} + \right. \\ \left.\dfrac{d_y u}{dy}\left(\dfrac{d_x v}{dx} + Q\right)\right] + ue^v\left(\dfrac{d_x d_y v}{dx\,dy} + Q\dfrac{d_y v}{dy} + R\right) + S. \end{array}\right\} = 0 \ldots (a_2)$$

2.° pone

$$\frac{d_x v}{dx}+Q=0\,,\ \frac{d_x d_y v}{dx dy}+\frac{d_x u}{dx}\left(\frac{d_y v}{dy}+P\right)=-\frac{S}{e^v}\ \ldots\ (a_4)\,;$$

ex (a_2) habebis

$$\frac{d_x d_y v}{dx dy}+P\frac{d_x v}{dx}+R=0\,,$$

seu, ob primam (a_4),

$$\frac{d_y Q}{dy}+PQ-R=0\ \ldots\ (a_5)\,:$$

hinc expleta (a_5), et determinatis v, u, integrale aequationis (a_1) erit $z=e^v u$. Binae v, u sic determinantur: prima (a_4) dat (200) $dy=0$, $dv+Qdx=0$; unde $y=C_1$, $v+\int Qdx=C_2$, et consequenter

$$v=-\int Qdx+\varphi(y)\,:$$

in secunda (a_4) fiat

$$\frac{d_x u}{dx}=u_1\,,$$

ut obtineatur

$$\frac{d_y u_1}{dy}+u_1\left(\frac{d_y v}{dy}+P\right)=-\frac{S}{e^v}\,;$$

ex istarum secunda eruemus (200) u_1, ex prima assequemur

$$u=\int u_1 dx+\psi(y).$$

3.° pone

$$\frac{d_y v}{dy}+P=0\,,\ \frac{d_x d_y u}{dx dy}+\frac{d_y u}{dy}\left(\frac{d_x v}{dx}+Q\right)=-\frac{S}{e^v}\ \ldots\ (a_6)\,;$$

ex (a_3) habebis

$$\frac{d_x d_y v}{dx dy}+Q\frac{d_y v}{dy}+R=0\,,$$

seu, ob primam (a_8),

$$\frac{d_x P}{dx} + PQ - R = 0 \ldots (a_7):$$

hinc expleta (a_7), integrale aequationis (a_1) erit iterum $z = e^v u$, determinatis v et u ex (a_8).

4.° aequatio (a_1) sic potest scribi

$$\frac{d_x\left(\frac{d_y z}{dy} + Pz\right)}{dx} + Q\left(\frac{d_y z}{dy} + Pz\right) + \left(R - \frac{d_x P}{dx} - PQ\right)z + S = 0;$$

quae, factis

$$\frac{d_y z}{dy} + Pz = z_1, \quad R - \frac{d_x P}{dx} - PQ = K \ldots (a_8),$$

traducetur ad

$$\frac{1}{K}\frac{d_x z_1}{dx} + \frac{Q}{K} z_1 + z + \frac{S}{K} = 0 \ldots (a_9):$$

et quia

$$K\frac{d_y(a_9)}{dy} = \frac{d_x d_y z_1}{dx dy} + K\frac{d_y \frac{1}{K}}{dy}\frac{d_x z_1}{dx} + Q\frac{d_y z_1}{dy} +$$

$$K\frac{d_y \frac{Q}{K}}{dy} z_1 + K\frac{d_y z}{dy} + K\frac{d_y \frac{S}{K}}{dy} = 0,$$

$$KP.(a_9) = P\frac{d_x z_1}{dx} + PQz_1 + KPz + S = 0;$$

ideo, factis compendii causa

$$K\frac{d_y \frac{1}{K}}{dy} + P = P_1, \quad K\frac{d_y \frac{Q}{K}}{dy} + PQ + K = R_1, \quad K\frac{d_y \frac{S}{K}}{dy} + S = S_1,$$

ob primam (a_8) exsistet

$$K\frac{d_y(a_9)}{dy}+KP.(a_9)=\frac{d_xd_yz_1}{dxdy}+P_1\frac{d_xz_1}{dx}+Q\frac{d_yz_1}{dy}+R_1z_1+S_1=0\ldots(a_{10}).$$

Quemadmodum (a_1) suppeditat (a_{10}), sic (a_{10}) praebebit

$$\frac{d_xd_yz_2}{dxdy}+P_2\frac{d_xz_2}{dx}+Q\frac{d_yz_2}{dy}+R_2z_2+S_2=0\ldots(a_{11});$$

atque ita porro.

5.° si neque (a_8) expletur, neque (a_7), expleatur autem altera e duabus

$$\left.\begin{array}{l}\dfrac{d_yQ}{dy}+P_1Q-R_1=0\,,\\[2ex] \dfrac{d_xP_1}{dx}+P_1Q-R_1=0\,,\end{array}\right\}(a_{12})$$

superiori methodo (2.° 3.°) determinabitur z_1, tum z ex prima (a_8): quod si neque prima (a_{12}) expleatur, neque secunda, perges ad

$$\left.\begin{array}{l}\dfrac{d_yQ}{dy}+P_2Q-R_2=0\,,\\[2ex] \dfrac{d_xP_2}{dx}+P_2Q-R_2=0\,;\end{array}\right\}(a_{13})$$

quarum altera expleta, determinabis z_2, tum z_1, denique z; et sic deinceps.

6.° ad integrationem aequationis (a_1) traduci potest integratio aequationis

$$\frac{d^2{}_xz}{dx^2}+f\frac{d_xd_yz}{dxdy}+f_1\frac{d^2{}_yz}{dy^2}+f_2\frac{d_xz}{dx}+f_3\frac{d_yz}{dy}+f_4z+f_5=0\ldots(a_{14}),$$

in qua f, f_1, ..., f_5 denotant functiones variabilium x, y. Sint enim r, s functiones earumdem x, y, specteturque z ut functio ipsarum r, s; erunt (48: 49)

$$\frac{d_x z}{dx}=\frac{d_r z}{dr}\frac{d_x r}{dx}+\frac{d_s z}{ds}\frac{d_x s}{dx}\ ,\ \frac{d_y z}{dy}=\frac{d_r z}{dr}\ \frac{d_y r}{dy}+\frac{d_s z}{ds}\ \frac{d_y s}{dy}\ ,$$

$$\frac{d^2{}_x z}{dx^2}=\frac{d^2{}_r z}{dr^2}\frac{d_x r^2}{dx^2}+\frac{d^2{}_s z}{ds^2}\frac{d_x s^2}{dx^2}+2\frac{d_r d_s z}{drds}\frac{d_x r}{dx}\frac{d_x s}{dx}+$$

$$\frac{d_r z}{dr}\ \frac{d^2{}_x r}{dx^2}+\frac{d_s z}{ds}\ \frac{d^2{}_x s}{dx^2}\ ,\ \frac{d^2{}_y z}{dy^2}=\frac{d^2{}_r z}{dr^2}\ \frac{d_y r^2}{dy^2}+$$

$$\frac{d^2{}_s z}{ds^2}\ \frac{d_y s^2}{dy^2}+2\frac{d_r d_s z}{drds}\ \frac{d_y r}{dy}\ \frac{d_y s}{dy}+\frac{d_r z}{dr}\ \frac{d^2{}_y r}{dy^2}+\frac{d_s z}{ds}\ \frac{d^2{}_y s}{dy^2}\ ,$$

$$\frac{d_x d_y z}{dxdy}=\frac{d^2{}_r z}{dr^2}\ \frac{d_x r}{dx}\ \frac{d_y r}{dy}+\frac{d^2{}_s z}{ds^2}\ \frac{d_x s}{dx}\ \frac{d_y s}{dy}+$$

$$\frac{d_r d_s z}{drds}\left(\frac{d_y r}{dy}\ \frac{d_x s}{dx}+\frac{d_x r}{dx}\ \frac{d_y s}{dy}\right)+\frac{d_r z}{dr}\ \frac{d_x d_y r}{dxdy}+\frac{d_s z}{ds}\ \frac{d_x d_y s}{dxdy}\ ;$$

adhibe substitutiones in (a_{14}), pone

$$\left.\begin{aligned}\frac{d_x r^2}{dx^2}+\mathrm{f}\frac{d_x r}{dx}\ \frac{d_y r}{dy}+\mathrm{f}_1\frac{d_y r^2}{dy^2}=0\ ,\\ \frac{d_x s^2}{dx^2}+\mathrm{f}\frac{d_x s}{dx}\ \frac{d_y s}{dy}+\mathrm{f}_1\frac{d_y s^2}{dy^2}=0\ ,\end{aligned}\right\}(a_{15})$$

itemque compendii causa

$$2\frac{d_x r}{dx}\ \frac{d_x s}{dx}+\mathrm{f}\left(\frac{d_x r}{dx}\ \frac{d_y s}{dy}+\frac{d_y r}{dy}\ \frac{d_x s}{dx}\right)+2\mathrm{f}_1\frac{d_y r}{dy}\ \frac{d_y s}{dy}=f_1\ ,$$

$$\frac{d^2{}_x r}{dx^2}+\mathrm{f}\frac{d_x d_y r}{dxdy}+\mathrm{f}_1\frac{d^2{}_y r}{dy^2}+\mathrm{f}_2\frac{d_x r}{dx}+\mathrm{f}_3\frac{d_y r}{dy}=f_2\ ,$$

$$\frac{d^2{}_x s}{dx^2}+\mathrm{f}\frac{d_x d_y s}{dxdy}+f_1\frac{d^2{}_y s}{dy^2}+\mathrm{f}_2\frac{d_x s}{dx}+\mathrm{f}_3\frac{d_y s}{dy}=f_3\ ;$$

traducetur (a_{14}) ad:

$$\frac{d_r d_s z}{dr\,ds}+\frac{f_2}{f_1}\frac{d_r z}{dr}+\frac{f_3}{f_1}\frac{d_s z}{ds}+\frac{f_4}{f_1}z+\frac{f_5}{f_1}=0 \ldots (a_{16}).$$

Determinatis (202) r et s ope aequationum (a_{15}), fiet gradus ad integrationem (2.° 3.° 4.° 5.°) aequationis (a_{16}).

205. Quod spectat ad integrationem differentialium partialiumque aequationum per series, tria subjicimus exempla.

I.° Sit

$$\frac{d_y z}{dy}=k\frac{d^2{}_x z}{dx^2}\ldots(a_{17}),$$

in qua exprimit k quantitatem constantem. Ac 1.° denotantibus Q_0, Q_1, Q_2,... functiones quantitatis y, et α_0, α_1, α_2, ... quantitates constantes, ponatur

$$z=Q_0 e^{\alpha_0 x}+Q_1 e^{\alpha_1 x}+Q_2 e^{\alpha_2 x}+\ldots;$$

provenient

$$\frac{d_y z}{dy}=Q'_0 e^{\alpha_0 x}+Q'_1 e^{\alpha_1 x}+Q'_2 e^{\alpha_2 x}+\ldots,$$

$$\frac{d^2{}_x z}{dx^2}=Q_0\alpha_0^2 e^{\alpha_0 x}+Q_1\alpha_1^2 e^{\alpha_1 x}+Q_2\alpha_2^2 e^{\alpha_2 x}+\ldots;$$

et adhibitis substitutionibus in (a_{17}),

$$Q'_0 e^{\alpha_0 x}+Q'_1 e^{\alpha_1 x}+Q'_2 e^{\alpha_2 x}+\ldots$$

$$=kQ_0\alpha_0^2 e^{\alpha_0 x}+kQ_1\alpha_1^2 e^{\alpha_1 x}+kQ_2\alpha_2^2 e^{\alpha_2 x}+\ldots.$$

Hinc

$$Q'_0=k\alpha_0^2 Q_0,\ Q'_1=k\alpha_1^2 Q_1,\ Q'_2=k\alpha_2^2 Q_2,\ldots,$$

ac proinde α_0, α_1, α_2, ... permanent arbitrariae; eruntque

$$Q_0 = c_0 e^{k\alpha_0^2 y},\ Q_1 = c_1 e^{k\alpha_1^2 y},\ Q_2 = c_2 e^{k\alpha_2^2 y},\ldots\ldots,$$

ubi denotant c_0, c_1, c_2, ... constantes atque arbitrarias quantitates. Integrale igitur aequationis (a_{17}) prodibit expressum per

$$z = c_0 e^{k\alpha_0^2 y} e^{\alpha_0 x} + c_1 e^{k\alpha_1^2 y} e^{\alpha_1 x} + c_2 e^{k\alpha_2^2 y} e^{\alpha_2 x} + \ldots (a_{18});$$

cujus praeterea singuli termini explebunt ipsam (a_{17}).

2.° formula (a_{18}) potest in hunc modum (241 ex p. 1^a.) scribi

$$z = c_0 e^{\alpha_0 x} + c_1 e^{\alpha_1 x} + c_2 e^{\alpha_2 x} + \ldots + (c_0 \alpha_0^2 e^{\alpha_0 x} + c_1 \alpha_1^2 e^{\alpha_1 x} + c_2 \alpha_2^2 e^{\alpha_2 x} + \ldots) ky + (c_0 \alpha_0^4 e^{\alpha_0 x} + c_1 \alpha_1^4 e^{\alpha_1 x} + c_2 \alpha_2^4 e^{\alpha_2 x} + \ldots) \frac{k^2 y^2}{1.2} + \ldots;$$

et assumpta functione arbitraria $\varphi(x)$, factoque

$$c_0 e^{\alpha_0 x} + c_1 e^{\alpha_1 x} + c_2 e^{\alpha_2 x} + \ldots = \varphi(x),$$

exsistent

$$c_0 \alpha_0^2 e^{\alpha_0 x} + c_1 \alpha_1^2 e^{\alpha_1 x} + c_2 \alpha_2^2 e^{\alpha_2 x} + \ldots = \varphi''(x),$$

$$c_0 \alpha_0^4 e^{\alpha_0 x} + c_1 \alpha_1^4 e^{\alpha_1 x} + c_2 \alpha_2^4 e^{\alpha_2 x} + \ldots = \varphi^{\mathrm{IV}}(x),$$

et caet. ...

vertetur nimirum (a_{18}) in

$$z = \varphi(x) + ky\varphi''(x) + \frac{k^2 y^2}{1.2} \varphi^{\mathrm{IV}}(x) + \ldots (a_{19}).$$

3°. designantibus C_0, C_1, C_2, ... constantes et arbitrarias quantitates, sit

$$\varphi(x) = C_0 + C_1 x + \frac{C_2 x^2}{1.2} + \frac{C_3 x^3}{1.2.3} + \ldots;$$

erunt

$$\varphi''(x)=C_2+C_3x+\frac{C_4x^2}{1.2}+\frac{C_5x^3}{1.2.3}+\ldots,$$

$$\varphi^{\mathrm{IV}}(x)=C_4+C_5x+\frac{C_6x^2}{1.2}+\frac{C_7x^3}{1.2.3}+\ldots,$$

et caet. . . .

Quare (a_{19}) immutabitur in

$$\left.\begin{array}{l} z=C_0+C_2ky+\dfrac{C_4k^2y^2}{1.2}+\ldots+(C_1+C_3ky+ \\ \dfrac{C_5k^2y^2}{1.2}+\ldots)x+(C_2+C_4ky+\dfrac{C_6k^2y^2}{1.2}+\ldots)\dfrac{x^2}{1.2}+ \\ (C_3+C_5ky+\dfrac{C_7k^2y^2}{1.2}+\ldots)\dfrac{x^3}{1.2.3}+\ldots \end{array}\right\}(a_{20}).$$

Sumptis arbitrariis functionibus $\varphi_1(y)$, $\varphi_2(y)$ invicem independentibus, pone

$$C_0+C_2ky+\frac{C_4k^2y^2}{1.2}+\ldots=\varphi_1(y),$$

$$C_1+C_3ky+\frac{C_5k^2y^2}{1.2}+\ldots=\varphi_2(y);$$

habebis

$$C_2+C_4ky+\ldots=\frac{1}{k}\varphi'_1(y),\ C_3+C_5ky+\ldots=\frac{1}{k}\varphi'_2(y),\text{et cet.};$$

et consequenter (a_{20}) poterit sic exprimi

$$\left.\begin{array}{l} z=\varphi_1(y)+\dfrac{x^2}{1.2k}\varphi'_1(y)+\dfrac{x^4}{1.2.3.4k^2}\varphi_1''(y)+\ldots+ \\ x\varphi_2(y)+\dfrac{x^3}{1.2.3k}\varphi'_2(y)+\dfrac{x^5}{1.2.3.4.5k^2}\varphi_2''(y)+\ldots \end{array}\right\}(a_{21})$$

4.° quisque nunc videt integrale aequationis (a_{17}) amplecti vel unam, vel duas arbitrarias functiones prout ordinatur vel secundum potentias quantitatis y, vel secundum potentias quantatis x: quod si ordinetur secundum potentias exponentialis e^x, nullam arbitrariam functionem explicite continebit, sed binas dumtaxat series infinitas arbitrariarum constantium c_0, c_1, c_2, et caet. α_0, α_1, α_2, et caet.

5.° possunt binae series (a_{18}), (a_{19}) exprimi sub forma finita per integralia definita; sed antequam id ostendamus, nonnulla subjungimus circa istiusmodi integralia. In formulis jam stabilitis (147) ponatur x pro a, et $-h$ pro h; emerget primo (162. 2.° ex p. 2.ª)

$$\left.\begin{aligned}&\int_0^\infty e^{-hx}(\cos kx+\sqrt{-1}\,\sin kx)dx=\\&-\frac{1}{-h+k\sqrt{-1}}=\frac{h+k\sqrt{-1}}{h^2+k^2};\end{aligned}\right\}(a_{20})$$

unde

$$\left.\begin{aligned}&\int_0^\infty e^{-hx}\cos kx\,dx=\frac{h}{h^2+k^2},\int_0^\infty e^{-hx}\sin kx\,dx=\frac{k}{h^2+k^2};\\&\text{in quarum prima facto prius } k=0\text{, ac dein}\\&k=0\text{, } h=1\text{, prodibunt}\\&\int_0^\infty e^{-hx}dx=\frac{1}{h}\text{, }\int_0^\infty e^{-x}dx=1:\end{aligned}\right\}(a_{21})$$

denotante α constantem arbitrariamque quantitatem >0, β vero numericum valorem vergentem, ad $lim.=0$, emerget secundo

$$\int_0^{\frac{1}{\beta\alpha}} x^n e^{-hx}(\cos kx + \sqrt{-1}\,\sin kx)dx =$$

$$\frac{\cos\frac{k}{\beta\alpha}+\sqrt{-1}\,\sin\frac{k}{\beta\alpha}}{\beta^n\alpha^n e^{\frac{h}{\beta\alpha}}(-h+k\sqrt{-1})}\Big[1-\frac{n\beta\alpha}{-h+k\sqrt{-1}}+\frac{n(n-1)\beta^2\alpha^2}{(-h+k\sqrt{-1})^2}$$

$$-\ldots\pm\frac{n(n-1)\ldots3.2.1.\beta^n\alpha^n}{(-h+k\sqrt{-1})^n}\Big]\mp\frac{n(n-1)\ldots3.2.1}{(-h+k\sqrt{-1})^{n+1}}.$$

Est autem (241 ex p. 1.ª)

$$\lim.\beta^n\alpha^n e^{\frac{h}{\beta\alpha}}=\lim.\beta^n\alpha^n\Big(1+\frac{h}{\beta\alpha}+\frac{h^2}{2\beta^2\alpha^2}+\frac{h^3}{2.3\beta^3\alpha^3}+\ldots$$

$$+\frac{h^n}{2.3\ldots n\beta^n\alpha^n}+\frac{h^{n+1}}{2.3\ldots n(n+1)\beta^{n+1}\alpha^{n+1}}+\ldots\Big)=\infty;$$

igitur

$$\int_0^{\infty} x^n e^{-hx}(\cos kx + \sqrt{-1}\,\sin kx)dx =$$

$$\mp\frac{n(n-1)\ldots3.2.1.(h+k\sqrt{-1})^{n+1}(-1)^{n+1}}{(h^2+k^2)^{n+1}}$$

$$=\frac{n(n-1)\ldots3.2.1.(h+k\sqrt{-1})^{n+1}}{(h^2+k^2)^{n+1}}:$$

et quoniam (157 3.° ex p. 2.ª)

$$(h+k\sqrt{-1})^{n+1}=(h^2+k^2)^{\frac{n+1}{2}}\left[\cos(n+1)\operatorname{arc}\left(\operatorname{tang}=\frac{k}{h}\right)+\right.$$
$$\left.\sqrt{-1}\,\sin(n+1)\operatorname{arc}\left(\operatorname{tang}=\frac{k}{h}\right)\right];$$

icc irco

$$\left.\begin{array}{l}\int_0^\infty x^n e^{-hx}(\cos kx+\sqrt{-1}\,\sin kx)dx= \\ \dfrac{n(n-1)\ldots3.2.1}{(h^2+k^2)^{\frac{n+1}{2}}}\left[\cos(n+1)\operatorname{arc}\left(\operatorname{tang}=\dfrac{k}{h}\right)+\right. \\ \qquad\left.\sqrt{-1}\,\sin(n+1)\operatorname{arc}\left(\operatorname{tang}=\dfrac{k}{h}\right)\right]\end{array}\right\}(a_{24})$$

unde

$$\left.\begin{array}{l}\int_0^\infty x^n e^{-hx}\cos kx\,dx=\dfrac{n(n-1)\ldots3.2.1}{(h^2+k^2)^{\frac{n+1}{2}}}\cos(n+1)\operatorname{arc}\left(\operatorname{tang}=\dfrac{k}{h}\right), \\ \int_0^\infty x^n e^{-hx}\sin kx\,dx=\dfrac{n(n-1)\ldots3.2.1}{(h^2+k^2)^{\frac{n+1}{2}}}\sin(n+1)\operatorname{arc}\left(\operatorname{tang}=\dfrac{k}{h}\right),\end{array}\right\}$$

in quarum prima facto prius $k=0$, ac dein $k=0$, $h=1$, provenient $a(_{25})$

$$\int_0^\infty x^n e^{-hx}dx=\frac{n(n-1)\ldots3.2.1}{h^{n+1}},$$

$$\int_0^\infty x^n e^{-x}dx=n(n-1)\ldots3.2.1.$$

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6.° si ponitur $hx=y$, erit

$$x^{\nu}e^{-hx}dx=\frac{y^{\nu}e^{-y}dy}{h^{\nu+1}}.$$

Hinc

$$\int_0^{\infty}x^{\nu}e^{-hx}dx=\frac{1}{h^{\nu+1}}\int_0^{\infty}x^{\nu}e^{-x}dx\ ;$$

et habito $\int_0^{\infty}x^{\nu}e^{-x}dx$ pro functione constantis arbitrariae ν, ut scribatur

$$\left.\begin{array}{l}\displaystyle\int_0^{\infty}x^{\nu}e^{-x}dx=\chi(\nu)\ ; \\ \text{erit} \\ \displaystyle\int_0^{\infty}x^{\nu}e^{-hx}dx=\frac{\chi(\nu)}{h^{\nu+1}}.\end{array}\right\}(a_{26})$$

Liquet, si ponitur successive

$\nu=0$, $\nu=1$, $\nu=2$, $\nu=3$, et caet ...,

fore ob postremas (a_{22}), (a_{25})

$\chi(0)=1$, $\chi(1)=1$, $\chi(2)=1.2$, $\chi(3)=1.2.3$, ...
$\chi(n)=1.2.3\ldots n$ (a_{27}).

7.° ex (a_{26}) manifeste eruitur

$$\int_0^{\infty}z^{\nu}e^{-z(x+1)}dz=\frac{\chi(\nu)}{(1+x)^{\nu+1}}\ ;$$

hinc

$$\frac{x^\alpha}{(1+x)^{\nu+1}}=\frac{1}{\chi(\nu)}\int_0^\infty x^\alpha e^{-zx} z^\nu e^{-z}\,dz,$$

et consequenter

$$\int_0^\infty \frac{x^\alpha dx}{(1+x)^{\nu+1}}=\frac{1}{\chi(\nu)}\int_0^\infty\int_0^\infty x^\alpha e^{-zx} z^\nu e^{-z}\,dx\,dz.$$

In secundo membro hujus aequationis sumptis integralibus prius quoad x, ac dein quoad z, ob (a[illegible]) prodibit

$$\left.\begin{aligned}\int_0^\infty \frac{x^\alpha dx}{(1+x)^{\nu+1}}=\frac{1}{\chi(\nu)}\int_0^\infty \frac{\chi(\alpha)}{z^{\alpha+1}} z^\nu e^{-z}\,dz=\\ \frac{\chi(\alpha)}{\chi(\nu)}\int_0^\infty z^{\nu-\alpha-1}e^{-z}\,dz=\frac{\chi(\alpha)\chi(\nu-\alpha-1)}{\chi(\nu)}\end{aligned}\right\}$$ (a[illegible])

8.° facto $\frac{n+1}{m}\pi=\theta$, satis erit recolere quae dicta sunt (71), necnon in p. 2.ª n.° 159. 1.° 3.° 4.°, ut intelligamus fore quoad m parem

$$\frac{x^n}{x^m-1}=\frac{1}{m}\Big[\frac{1}{x-1}+\frac{\cos 2\theta-\sqrt{-1}\sin 2\theta}{x-\cos\frac{2\pi}{m}+\sqrt{-1}\sin\frac{2\pi}{m}}+$$

$$\frac{\cos 2\theta+\sqrt{-1}\sin 2\theta}{x-\cos\frac{2\pi}{m}-\sqrt{-1}\sin\frac{2\pi}{m}}+\frac{\cos 4\theta-\sqrt{-1}\sin 4\theta}{x-\cos\frac{4\pi}{m}+\sqrt{-1}\sin\frac{4\pi}{m}}+$$

$$\frac{\cos 4\theta+\sqrt{-1}\sin 4\theta}{x-\cos\frac{4\pi}{m}-\sqrt{-1}\sin\frac{4\pi}{m}}+\ldots+$$

$$\frac{\cos(m-2)\theta-\sqrt{-1}\,\sin(m-2)\theta}{x-\cos\frac{m-2}{m}\pi+\sqrt{-1}\,\sin\frac{m-2}{m}\pi}+$$

$$\frac{\cos(m-2)\theta+\sqrt{-1}\,\sin(m-2)\theta}{x-\cos\frac{m-2}{m}\pi-\sqrt{-1}\,\sin\frac{m-2}{m}\pi}-\frac{(-1)^m}{x+1}],\quad \frac{x^n}{x^m+1}=$$

$$-\frac{1}{m}[\frac{\cos\theta-\sqrt{-1}\,\sin\theta}{x-\cos\frac{\pi}{m}+\sqrt{-1}\,\sin\frac{\pi}{m}}+\frac{\cos\theta+\sqrt{-1}\,\sin\theta}{x-\cos\frac{\pi}{m}-\sqrt{-1}\,\sin\frac{\pi}{m}}+$$

$$\frac{\cos 3\theta-\sqrt{-1}\,\sin 3\theta}{x-\cos\frac{3\pi}{m}+\sqrt{-1}\,\sin\frac{3\pi}{m}}+\frac{\cos 3\theta+\sqrt{-1}\,\sin 3\theta}{x-\cos\frac{3\pi}{m}-\sqrt{-1}\,\sin\frac{3\pi}{m}}+\dots$$

$$+\frac{\cos(m-1)\theta-\sqrt{-1}\,\sin(m-1)\theta}{x-\cos\frac{m-1}{m}\pi+\sqrt{-1}\,\sin\frac{m-1}{m}\pi}+$$

$$\frac{\cos(m-1)\theta+\sqrt{-1}\,\sin(m-1)\theta}{x-\cos\frac{m-1}{m}\pi-\sqrt{-1}\,\sin\frac{m-1}{m}\pi}];$$

quoad m imparem

$$\frac{x^n}{x^m-1}=\frac{1}{m}[\frac{1}{x-1}+\frac{\cos 2\theta-\sqrt{-1}\,\sin 2\theta}{x-\cos\frac{2\pi}{m}+\sqrt{-1}\,\sin\frac{2\pi}{m}}+$$

$$\frac{\cos 2\theta+\sqrt{-1}\sin 2\theta}{x-\cos\frac{2\pi}{m}-\sqrt{-1}\sin\frac{2\pi}{m}}+\frac{\cos 4\theta-\sqrt{-1}\sin 4\theta}{x-\cos\frac{4\pi}{m}+\sqrt{-1}\sin\frac{4\pi}{m}}+$$

$$\frac{\cos 4\theta+\sqrt{-1}\sin 4\theta}{x-\cos\frac{4\pi}{m}-\sqrt{-1}\sin\frac{4\pi}{m}}+\ldots+\frac{\cos(m-1)\theta-\sqrt{-1}\sin(m-1)\theta}{x-\cos\frac{m-1}{m}\pi+\sqrt{-1}\sin\frac{m-1}{m}\pi}$$

$$+\frac{\cos(m-1)\theta+\sqrt{-1}\sin(m-1)\theta}{x-\cos\frac{m-1}{m}\pi-\sqrt{-1}\sin\frac{m-1}{m}\pi}],\ \frac{x^n}{x^m+1}=$$

$$-\frac{1}{m}[\frac{\cos\theta-\sqrt{-1}\sin\theta}{x-\cos\frac{\pi}{m}+\sqrt{-1}\sin\frac{\pi}{m}}+\frac{\cos\theta+\sqrt{-1}\sin\theta}{x-\cos\frac{\pi}{m}-\sqrt{-1}\sin\frac{\pi}{m}}+$$

$$\frac{\cos 3\theta-\sqrt{-1}\sin 3\theta}{x-\cos\frac{3\pi}{m}+\sqrt{-1}\sin\frac{3\pi}{m}}+\frac{\cos 3\theta+\sqrt{-1}\sin 3\theta}{x-\cos\frac{3\pi}{m}-\sqrt{-1}\sin\frac{3\pi}{m}}+\ldots$$

$$+\frac{\cos(m-2)\theta-\sqrt{-1}\sin(m-2)\theta}{x-\cos\frac{m-2}{m}\pi+\sqrt{-1}\sin\frac{m-2}{m}\pi}+$$

$$\frac{\cos(m-2)\theta+\sqrt{-1}\sin(m-2)\theta}{x-\cos\frac{m-2}{m}\pi-\sqrt{-1}\sin\frac{m-2}{m}\pi}-\frac{(-1)^n}{x+1}].$$

Quare (137) pro m pari

$$\int\frac{x^n dx}{x^m-1}=\frac{1}{m}[\cos\frac{2(n+1)}{m}\pi L(x^2-2x\cos\frac{2\pi}{m}+1)+$$

$$\cos\frac{4(n+1)}{m}\pi \mathrm{L}(x^2-2x\cos\frac{4\pi}{m}+1)+\dots+\cos\frac{(m-2)(n+1)}{m}\pi \mathrm{L}(x^2-$$

$$2x\cos\frac{m-2}{m}\pi+1)]-\frac{2}{m}[\sin\frac{2(n+1)}{m}\pi.\mathrm{arc}(\mathrm{tang}=\frac{x-\cos\frac{2\pi}{m}}{\sin\frac{2\pi}{m}})+$$

$$\sin\frac{4(n+1)}{m}\pi.\mathrm{arc}(\mathrm{tang}=\frac{x-\cos\frac{4\pi}{m}}{\sin\frac{4\pi}{m}})+\dots+\sin\frac{(m-2)(n+1)}{m}\pi.\mathrm{arc}(\mathrm{tang}=$$

$$\frac{x-\cos\frac{m-2}{m}\pi}{\sin\frac{m-2}{m}\pi})]+\frac{1}{2m}\mathrm{L}(x-1)^2-\frac{(-1)^n}{2m}\mathrm{L}(x+1)^2+C,$$

$$\int\frac{x^n dx}{x^m+1}=-\frac{1}{m}[\cos\frac{n+1}{m}\pi \mathrm{L}(x^2-2x\cos\frac{\pi}{m}+1)+$$

$$\cos\frac{3(n+1)}{m}\pi \mathrm{L}(x^2-2x\cos\frac{3\pi}{m}+1)+\dots+\cos\frac{(m-1)(n+1)}{m}\pi \mathrm{L}(x^2-$$

$$2x\cos\frac{m-1}{m}\pi+1)]+\frac{2}{m}[\sin\frac{n+1}{m}\pi\, \mathrm{arc}(\mathrm{tang}=\frac{x-\cos\frac{\pi}{m}}{\sin\frac{\pi}{m}})+$$

$$\sin\frac{(3n+1)}{m}\pi.\mathrm{arc}(\mathrm{tang}=\frac{x-\cos\frac{3\pi}{m}}{\sin\frac{3\pi}{m}})+\dots+\sin\frac{(m-1)(n+1)}{m}\pi\times$$

$$arc\left(tang = \frac{x - cos\frac{m-1}{m}\pi}{sin\frac{m-1}{m}\pi}\right)\Big] + C =$$

et pro m impari

$$\int\frac{x^n dx}{x^m - 1} = \frac{1}{m}\Big[cos\frac{2(n+1)}{m}\pi L(x^2 - 2x\,cos\frac{2\pi}{m} + 1) + cos\frac{4(n+1)}{m}\pi L(x^2 - 2x\,cos\frac{4\pi}{m} + 1) + \ldots + cos\frac{(m-1)(n+1)}{m}\pi (Lx^2 - 2x\,cos\frac{m-1}{m}\pi + 1)\Big] - \frac{2}{m}\Big[sin\frac{2(n+1)}{m}\pi . arc\left(tang = \frac{x - cos\frac{2\pi}{m}}{sin\frac{2\pi}{m}}\right) + sin\frac{4(n+1)}{m}\pi . arc\left(tang = \frac{x - cos\frac{4\pi}{m}}{sin\frac{4\pi}{m}}\right) + \ldots + sin\frac{(m-1)(n+1)}{m}\pi . arc\left(tang = \frac{x - cos\frac{m-1}{m}\pi}{sin\frac{m-1}{m}\pi}\right)\Big] + \frac{1}{2m}L(x-1)^2 + C,$$

$$\int\frac{x^n dx}{x^m + 1} = -\frac{1}{m}\Big[cos\frac{n+1}{m}\pi L(x^2 - 2x\,cos\frac{\pi}{m} + 1) + cos\frac{3(n+1)}{m}\pi L(x^2 - 2x\,cos\frac{3\pi}{m} + 1) + \ldots + cos\frac{(m-2)(n+1)}{m}\pi L(x^2 - 2x\,cos\frac{m-2}{m}\pi + 1)\Big] + \frac{2}{m}\Big[sin\frac{n+1}{m}\pi \times$$

$$\left. arc\left(tang = \frac{x - \cos\frac{\pi}{m}}{\sin\frac{\pi}{m}}\right) + \sin\frac{3(n+1)}{m}\pi . arc\left(tang = \right.\right.$$

$$\left.\frac{x - \cos\frac{3\pi}{m}}{\sin\frac{3\pi}{m}}\right) + \ldots + \sin\frac{(m-2)(n+1)}{m}\pi . arc\left(tang = \right.$$

$$\left.\left.\frac{x - \cos\frac{m-2}{m}\pi}{\sin\frac{m-2}{m}\pi}\right)\right] + \frac{(-1)^n}{2m} L(1+x)^2 + C.$$

Hinc, praeter α (5.°) assumpta et altera constanti arbitrariaque quantitate $\alpha' > 0$, erit v. gr.

$$\int_{-\infty}^{\infty} \frac{x^{2n}dx}{x^{2m}+1} = lim. \int_{-\frac{1}{\alpha'\beta}}^{\frac{1}{\alpha\beta}} \frac{x^{2n}}{x^{2m}+1} = - \frac{1}{m}\left[\cos\frac{2n+1}{2m}\pi + \right.$$

$$\left.\cos\frac{3(2n+1)}{2m}\pi + \ldots + \cos\frac{(2m-1)(2n+1)}{2m}\pi\right] L\left(\frac{\alpha'}{\alpha}\right) +$$

$$\frac{\pi}{m}\left[\sin\frac{2n+1}{2m}\pi + \sin\frac{3(2n+1)}{2m}\pi + \ldots + \sin\frac{(2m-1)(2n+1)}{2m}\pi\right].$$

Est autem (162. 2.° ex p. 2.ª)

$$\cos\frac{2n+1}{2m}\pi + \cos\frac{3(2n+1)}{2m}\pi + \ldots + \cos\frac{(2m-1)(2n+1)}{2m}\pi =$$

$$\tfrac{1}{2}\Big[e^{\frac{2n+1}{2m}\pi\sqrt{-1}}+e^{\frac{3(2n+1)}{2m}\pi\sqrt{-1}}+\dots+e^{\frac{(2m-1)(2n+1)}{2m}\pi\sqrt{-1}}+$$

$$e^{-\frac{2n+1}{2m}\pi\sqrt{-1}}+e^{-\frac{3(2n+1)}{2m}\pi\sqrt{-1}}+\dots+e^{-\frac{(2m-1)(2n+1)}{2m}\pi\sqrt{-1}}\Big]$$

$$=\frac{e^{\frac{2n+1}{2m}\pi\sqrt{-1}}}{2}\left(\frac{e^{(2n+1)\pi\sqrt{-1}}-1}{e^{\frac{2n+1}{m}\pi\sqrt{-1}}-1}\right)+$$

$$\frac{e^{-\frac{2n+1}{2m}\pi\sqrt{-1}}}{2}\left(\frac{e^{-(2n+1)\pi\sqrt{-1}}-1}{e^{-\frac{2n+1}{m}\pi\sqrt{-1}}-1}\right)$$

$$=\frac{-\left(e^{\frac{2n+1}{2m}\pi\sqrt{-1}}-e^{-\frac{2n+1}{2m}\pi\sqrt{-1}}\right)\left(e^{(2n+1)\pi\sqrt{-1}}-e^{-(2n+1)\pi\sqrt{-1}}\right)}{2\left(2-e^{\frac{2n+1}{m}\pi\sqrt{-1}}-e^{-\frac{2n+1}{m}\pi\sqrt{-1}}\right)}$$

$$=\frac{\sin\frac{2n+1}{2m}\pi.\sin(2n+1)\pi}{1-\cos\frac{2n+1}{m}\pi}=0$$: insuper

$$\sin\frac{2n+1}{2m}\pi+\sin\frac{3(2n+1)}{2m}\pi+\dots+\sin\frac{(2m-1)(2n+1)}{2m}\pi=$$

$$\frac{1}{2\sqrt{-1}}\Big[e^{\frac{2n+1}{2m}\pi\sqrt{-1}}+e^{\frac{3(2n+1)}{2m}\pi\sqrt{-1}}+\dots+e^{\frac{(2m-1)(2n+1)}{2m}\pi\sqrt{-1}}-$$

$$e^{-\frac{2n+1}{2m}\pi\sqrt{-1}}-e^{-\frac{3(2n+1)}{2m}\pi\sqrt{-1}}-\dots-e^{-\frac{(2m-1)(2n+1)}{2m}\pi\sqrt{-1}}\Big]=$$

$$\frac{e^{\frac{2n+1}{2m}\pi\sqrt{-1}}}{2\sqrt{-1}}\left(\frac{e^{(2n+1)\pi\sqrt{-1}}-1}{e^{\frac{2n+1}{m}\pi\sqrt{-1}}-1}\right)-\frac{e^{-\frac{2n+1}{2m}\pi\sqrt{-1}}}{2\sqrt{-1}}\left(\frac{e^{-(2n+1)\pi\sqrt{-1}}-1}{e^{-\frac{2n+1}{m}\pi\sqrt{-1}}-1}\right)$$

$$=\frac{\left(e^{\frac{2n+1}{2m}\pi\sqrt{-1}}-e^{-\frac{2n+1}{2m}\pi\sqrt{-1}}\right)\left(2-e^{(2n+1)\pi\sqrt{-1}}-e^{-(2n+1)\pi\sqrt{-1}}\right)}{2\sqrt{-1}\left(2-e^{\frac{2n+1}{m}\pi\sqrt{-1}}-e^{-\frac{2n+1}{m}\pi\sqrt{-1}}\right)}$$

$$=\frac{\sin\frac{2n+1}{2m}\pi\,(1-\cos(2n+1)\pi)}{1-\cos\frac{2n+1}{m}\pi}=\frac{\sin^2\frac{1}{2}(2n+1)\pi}{\sin\frac{2n+1}{2m}\pi}=\frac{1}{\sin\frac{2n+1}{2m}\pi};$$

ergo

$$\int_{-\infty}^{\infty}\frac{x^{2n}dx}{x^{2m}+1}=\frac{\pi}{m\sin\frac{2n+1}{2m}\pi}\quad\dots\dots(\alpha_{29}).$$

9.° facto $x^{2m}=z$, erit

$$\frac{x^{2n}dx}{x^{2m}+1}=\frac{1}{2m}\cdot\frac{z^{\frac{2n+1}{2m}-1}dz}{z+1}.$$

Ad haec: ex dictis (131 : 129 : 127) eruitur

$$\int_0^x\frac{x^{2n}dx}{x^{2m}+1}=-\int_0^{-x}\frac{x^{2n}dx}{x^{2m}+1}=\int_{-x}^0\frac{x^{2n}dx}{x^{2m}+1};$$

itemque (128)

$$\int_{-x}^x\frac{x^{2n}dx}{x^{2m}+1}=\int_{-x}^0\frac{x^{2n}dx}{x^{2m}+1}+\int_0^x\frac{x^{2n}dx}{x^{2m}+1}:$$

hinc

$$\int_0^x\frac{x^{2n}dx}{x^{2m}+1}=\tfrac{1}{2}\int_{-x}^x\frac{x^{2n}dx}{x^{2m}+1}.$$

Quibus positis, manifeste exurgit

$$\int_0^\infty\frac{z^{\frac{2n+1}{2m}-1}dz}{z+1}=2m\int_0^\infty\frac{x^{2n}dx}{x^{2m}+1}=$$

$$m\int_{-\infty}^\infty\frac{x^{2n}dx}{x^{2m}+1}=\frac{\pi}{\sin\frac{2n+1}{2m}\pi}\ldots(a_{30})$$

10.° facile quoque pervenitur ad

$$\int_{-\infty}^\infty\frac{x^{2n}dx}{1-x^{2m}}=-\int_{-\infty}^\infty\frac{x^{2n}dx}{x^{2m}-1}=-\frac{1}{m}(\cos\frac{2n+1}{m}\pi+$$

$$\cos\frac{2(2n+1)}{m}\pi+\cos\frac{3(2n+1)}{m}\pi+\ldots+\cos\frac{(m-1)(2n+1)}{m}\pi)L(\frac{\alpha'}{\alpha})+$$

$$\frac{\pi}{m}(\sin\frac{2n+1}{m}\pi+\sin\frac{2(2n+1)}{m}\pi+\sin\frac{3(2n+1)}{m}\pi+\ldots.$$

$$+\sin\frac{(m-1)(2n+1)}{m}\pi)-\frac{1}{2m}L(\frac{\alpha'}{\alpha})+\frac{1}{2m}L(\frac{\alpha'}{\alpha}).$$

Jam vero eadem methodo, qua supra (8.°) usi sumus, inveniemus

$$\cos\frac{2n+1}{m}\pi+\cos\frac{2(2n+1)}{m}\pi+\cos\frac{3(2n+1)}{m}\pi+\ldots+\cos\frac{(m-1)(2n+1)}{m}\pi=$$

$$\frac{\cos\frac{(m-1)(2n+1)}{m}\pi+\cos\frac{2n+1}{m}\pi}{2(1-\cos\frac{2n+1}{m}\pi)}=$$

$$\frac{\cos\frac{2n+1}{2}\pi\cos\frac{(m-2)(2n+1)}{2m}\pi}{2\sin^2\frac{2n+1}{2m}\pi}=0,$$

$$\sin\frac{2n+1}{m}\pi+\sin\frac{2(2n+1)}{m}\pi+\sin\frac{3(2n+1)}{m}\pi+\ldots+\sin\frac{(m-1)(2n+1)}{m}\pi=$$

$$\frac{\sin\frac{(m-1)(2n+1)}{m}\pi-\sin(2n+1)\pi+\sin\frac{2n+1}{m}\pi}{2(1-\cos\frac{2n+1}{m}\pi)}=$$

$$\frac{\sin\left((2n+1)\pi - \frac{2n+1}{m}\pi\right) + \sin\frac{2n+1}{m}\pi}{2\left(1-\cos\frac{2n+1}{m}\pi\right)} =$$

$$\frac{\sin\frac{2n+1}{m}\pi}{2\sin^2\frac{2n+1}{2m}\pi} = \frac{\cos\frac{2n+1}{2m}\pi}{\sin\frac{2n+1}{2m}\pi} = \frac{1}{\tang\frac{2n+1}{2m}\pi}.$$

Itaque

$$\int_{-\infty}^{\infty}\frac{x^{2n}dx}{1-x^{2m}} = \frac{\pi}{m\tang\frac{2n+1}{2m}\pi}\ldots(a_{31});$$

et facto $x^{2m}=z$, ut sit

$$\frac{x^{2n}dx}{1-x^{2m}} = \frac{1}{2m}\cdot\frac{z^{\frac{2n+1}{2m}-1}dz}{1-z},$$

prodibit (9.°)

$$\int_{0}^{\infty}\frac{z^{\frac{2n+1}{2m}-1}dz}{1-z} = 2m\int_{0}^{\infty}\frac{x^{2n}dx}{1-x^{2m}} =$$

$$m\int_{-\infty}^{\infty}\frac{x^{2n}dx}{1-x^{2m}} = \frac{\pi}{\tang\frac{2n+1}{2m}\pi}\ldots(a_{32}).$$

11.° in (a_{28}. 7.°) fiant $\nu=0$, $\alpha=\frac{2n+1}{2m}-1$: ob (a_{30}. 9.°), et ob primam (a_{27}. 6.°) erit

$$\left.\begin{array}{l}\chi\left(\frac{2n+1}{2m}-1\right).\chi\left(-\frac{2n+1}{2m}\right)=\frac{\pi}{\sin\frac{2n+1}{2m}\pi}\,; \\ \text{et assumptis } n=0\,,\ m=1\,, \\ \qquad \chi^2(-\tfrac{1}{2})=\pi\,,\ \chi(-\tfrac{1}{2})=\pi^{\frac{1}{2}}.\end{array}\right\}\ (a_{33})$$

Est autem (6.° a_{26})

$$\chi(-\tfrac{1}{2})=\int_0^\infty x^{-\frac{1}{2}}e^{-x}dx\,;$$

insuper facto prius $x=(z+\theta)^2$, dein $x=(z-\theta)^2$, habebimus

$$\int_0^\infty x^{-\frac{1}{2}}e^{-x}dx=2\int_0^\infty e^{-(z+\theta)^2}dz\,,$$

$$\int_0^\infty x^{-\frac{1}{2}}e^{-x}dx=2\int_0^\infty e^{-(z-\theta)^2}dz.$$

Itaque

$$\int_0^\infty x^{-\frac{1}{2}}e^{-x}dx=\pi^{\frac{1}{2}}=e^{-\theta^2}\int_0^\infty e^{-z^2}(e^{-2\theta z}+e^{2\theta z})dz:$$

et consequenter

$$\int_0^\infty e^{-z^2}(e^{2\theta z}+e^{-2\theta z})dz=\pi^{\frac{1}{2}}e^{\theta^2}\ \ldots\ (a_{34}).$$

Ad haec: si in (a_{34}) adhibetur $x\sqrt{-1}$ pro θ, exsurget (162. 2.° ex p. 2.^{a})

$$\int_0^\infty e^{-z^2}\cos 2xz\,dz=\tfrac{1}{2}\pi^{\frac{1}{2}}e^{-x^2};$$

quod si fiat $\theta=0$, proveniet

$$\int_0^\infty e^{-z^2}dz=\tfrac{1}{2}\pi^{\frac{1}{2}}: \qquad (a_{35})$$

et substituto $\alpha^{\frac{1}{2}}z$ pro z,

$$\int_0^\infty e^{-\alpha z^2}dz=\tfrac{1}{2}\left(\frac{\pi}{\alpha}\right)^{\frac{1}{2}}.$$

12.° in postrema (a_{35}) sumatur derivata n^{sima} quoad α; erit (134)

$$\int_0^\infty z^{2n}e^{-\alpha z^2}dz=\frac{1.3.5...(2n-1)}{2^{n+1}}\pi^{\frac{1}{2}}\alpha^{-\frac{2n+1}{2}};$$

et facta $\alpha=1$, $\qquad (a_{36})$

$$\int_0^\infty z^{2n}e^{-z^2}dz=\frac{1.3.5...(2n-1)}{2^{n+1}}\pi^{\frac{1}{2}}.$$

13.° habemus (131 : 129 : 127)

$$\int_0^z e^{-z^2}dz=-\int_0^{-z}e^{-z^2}dz=\int_{-z}^0 e^{-z^2}dz,$$

$$\int_0^z z^{2n}e^{-z^2}dz=-\int_0^{-z}z^{2n}e^{-z^2}dz=\int_{-z}^0 z^{2n}e^{-z^2}dz;$$

e secunda igitur (a_{35}) profluet (128)

$$\left.\begin{array}{l}\displaystyle\int_{-\infty}^{\infty} e^{-z^2}dz=\pi^{\frac{1}{2}},\\ \text{et e secunda } (a_{26})\\ \displaystyle\int_{-\infty}^{\infty} z^{2n}e^{-z^2}dz=\frac{1.3.5...(2n-1)}{2^n}\pi^{\frac{1}{2}}:\end{array}\right\}(a_{27})$$

est insuper (131 : 129 : 127)

$$\int_0^z z^{2n-1}e^{-z^2}dz=\int_0^{-z} z^{2n-1}e^{-z^2}dz=-\int_{-z}^0 z^{2n-1}e^{-z^2}dz;$$

ideo (128)

$$\int_{-\infty}^{\infty} z^{2n-1}e^{-z^2}dz=0\ldots(a_{28}).$$

14.° traducamus nunc seriem (a_{19}) ad integralia definita : prima (a_{27}) suppeditat

$$\varphi(x)=\frac{1}{\pi^{\frac{1}{2}}}\varphi(x)\int_{-\infty}^{\infty} e^{-r^2}dr=\frac{1}{\pi^{\frac{1}{2}}}\int_{-\infty}^{\infty}\varphi(x)e^{-r^2}dr;$$

secunda (a_{27}) praebet

$$k\gamma\varphi''(x)=\frac{2}{\pi^{\frac{1}{2}}}k\gamma\varphi''(x)\int_{-\infty}^{\infty} r^2e^{-r^2}dr=$$

$$\frac{1}{\pi^{\frac{1}{2}}}\int_{-\infty}^{\infty}\frac{4r^2k\gamma}{1.2}\varphi''(x)e^{-r^2}dr,\ \frac{k^2\gamma^2}{1.2}\varphi^{\text{IV}}(x)=$$

$$\frac{4}{3\pi^{\frac{1}{2}}}\,\frac{k^2\gamma^2}{1.2}\varphi^{\text{IV}}(x)\int_{-\infty}^{\infty} r^4e^{-r^2}dr=$$

$$\frac{1}{\pi^{\frac{1}{2}}}\int_{-\infty}^{\infty}\frac{16r^4k^2y^2}{1.2.3.4}\varphi^{IV}(x)\,e^{-r^2}dr\text{, et caet. ....;}$$

formula (a_{38}) dat

$$\frac{1}{\pi^{\frac{1}{2}}}\int_{-\infty}^{\infty}2r(ky)^{\frac{1}{2}}\varphi'(x)e^{-r^2}dr=0\,,$$

$$\frac{1}{\pi^{\frac{1}{2}}}\int_{-\infty}^{\infty}\frac{8r^3(ky)^{\frac{3}{2}}}{1.2.3}\varphi'''(x)e^{-r^2}dr=0\text{, et caet. ....:}$$

icc irco (128) poterit (a_{29}) scribi in hunc modum

$$z=\frac{1}{\pi^{\frac{1}{2}}}\int_{-\infty}^{\infty}[\varphi(x)+\frac{2r(ky)^{\frac{1}{2}}}{1}\varphi'(x)+\frac{4r^2(xy)}{1.2}\varphi''(x)+$$

$$\frac{8r^3(ky)^{\frac{3}{2}}}{1.2.3}\varphi'''(x)+\frac{16r^4(ky)^2}{1.2.3.4}\varphi^{IV}(x)+\ldots]\,e^{-r^2}dr\,;$$

et consequenter (32. b^{IV})

$$z=\frac{1}{\pi^{\frac{1}{2}}}\int_{-\infty}^{\infty}e^{-r^2}\varphi(x+2r\sqrt{ky})dr\ldots(a_{39}).$$

15.° ad (a_{28}) quod spectat, ex prima (a_{27}) assequimur

$$\int_{-\infty}^{\infty}e^{-(r-\alpha_0(ky)^{\frac{1}{2}})^2}dr=\pi^{\frac{1}{2}},\int_{-\infty}^{\infty}e^{-(r-\alpha_1(ky)^{\frac{1}{2}})^2}dr=\pi^{\frac{1}{2}},\text{et caet....}$$

unde

$$e^{\alpha_0^2 ky} = \frac{1}{\pi^{\frac{1}{2}}} \int_{-\infty}^{\infty} e^{-r^2} e^{2\alpha_0 r (ky)^{\frac{1}{2}}} dr,$$

$$e^{\alpha_1^2 ky} = \frac{1}{\pi^{\frac{1}{2}}} \int_{-\infty}^{\infty} e^{-r^2} e^{2\alpha_1 r (ky)^{\frac{1}{2}}} dr, \text{ et caet.} \ldots;$$

poteritque (a_{18}) sic exprimi

$$z = \frac{1}{\pi^{\frac{1}{2}}} \int_{-\infty}^{\infty} [c_0 e^{\alpha_0 (x + 2r(ky)^{\frac{1}{2}})} + c_1 e^{\alpha_1 (x + 2r(ky)^{\frac{1}{2}})} + \ldots] e^{-r^2} dr:$$

et posito ut supra (2.°)

$$c_0 e^{\alpha_0 x} + c_1 e^{\alpha_1 x} + c_2 e^{\alpha_2 x} + \ldots = \varphi(x),$$

ac per consequens

$$c_0 e^{\alpha_0 (x + 2r(ky)^{\frac{1}{2}})} + c_1 e^{\alpha_1 (x + 2r(ky)^{\frac{1}{2}})} + \ldots = \varphi(x + 2r(ky)^{\frac{1}{2}}),$$

prodibit iterum (a_{39}). Sed tempus est jam de aliis exemplis aliquid dicere.

II.° Sit

$$\frac{d^2{}_x z}{dx^2} + \frac{d^2{}_y z}{dy^2} + \frac{d^2{}_t z}{dt^2} = 0 \ldots (a_{40}).$$

Denotantibus M, N, P, Q, R, S, . . . functiones binarum x, y, sume

$$z = \mathrm{M} + \mathrm{N}t + \mathrm{P}t^2 + \mathrm{Q}t^3 + \mathrm{R}t^4 + \mathrm{S}t^5 + \ldots;$$

erunt

$$\frac{d^2{}_x z}{dx^2} = \mathrm{M}''_x + \mathrm{N}''_x t + \mathrm{P}''_x t^2 + \mathrm{Q}''_x t^3 + \ldots,$$

$$\frac{d^2{}_y z}{dy^2} = \mathrm{M}''_y + \mathrm{N}''_y t + \mathrm{P}''_y t^2 + \mathrm{Q}''_y t^3 + \ldots,$$

$$\frac{d_t^2 z}{dt^2} = 2P + 3.2Qt + 4.3Rt^2 + 5.4St^3 + \ldots$$

et adhibitis substitutionibus in (a_{40}),

$$M''_x + M''_y + 2P + (N''_x + N''_y + 3.2Q)t +$$
$$(P''_x + P''_y + 4.3R)t^2 + (Q''_x + Q''_y + 5.4S)t^3 + \ldots = 0.$$

Hinc

$$P = -\tfrac{1}{2}(M''_x + M''_y) = -\tfrac{1}{2}\left(\frac{d^2{}_x M}{dx^2} + \frac{d^2{}_y M}{dy^2}\right),$$

$$Q = -\frac{1}{3.2}(N''_x + N''_y) = -\frac{1}{3.2}\left(\frac{d^2{}_x N}{dx^2} + \frac{d^2{}_y N}{dy^2}\right),$$

$$R = -\frac{1}{4.3}(P''_x + P''_y) = \frac{1}{4.3.2}\left(\frac{d^4{}_x M}{dx^4} + 2\frac{d^2{}_x d^2{}_y M}{dx^2 dy^2} + \frac{d^4{}_y M}{dy^4}\right),$$

et caet. . . . ; et consequenter

$$z = M + Nt - \tfrac{1}{2}\left(\frac{d^2{}_x M}{dx^2} + \frac{d^2{}_y M}{dy^2}\right)t^2 - \frac{1}{3.2}\left(\frac{d^2{}_x N}{dx^2} + \frac{d^2{}_y N}{dy^2}\right)t^3 + \ldots (a_{41}):$$

binas dumtaxat arbitrarias functiones complectitur formula (a_{41})

Si poneretur (I.°)

$$z = Me^{\alpha_0 t} + Ne^{\alpha_1 t} + Pe^{\alpha_2 t} + \ldots,$$

forent

$$\frac{d^2{}_x z}{dx^2} = M''_x e^{\alpha_0 t} + N''_x e^{\alpha_1 t} + P''_x e^{\alpha_2 t} + \ldots,$$

$$\frac{d^2{}_y z}{dy^2} = M''_y e^{\alpha_0 t} + N''_y e^{\alpha_1 t} + P''_y e^{\alpha_2 t} + \ldots,$$

$$\frac{d_t^2 z}{dt^2} = \mathrm{M}\alpha^2{}_0 e^{\alpha_0 t} + \mathrm{N}\alpha^2{}_1 e^{\alpha_1 t} + \mathrm{P}\alpha^2{}_2 e^{\alpha_2 t} + \ldots ;$$

ideoque

$$(\mathrm{M}''_x + \mathrm{M}''_y + \mathrm{M}\alpha^2{}_0)e^{\alpha_0 t} + (\mathrm{N}''_x + \mathrm{N}''_y + \mathrm{N}\alpha^2{}_1)e^{\alpha_1 t} + \ldots = 0.$$

Ad functiones videlicet M, N, ... determinandas haberentur differentiales partialesque aequationes

$$\mathrm{M}''_x + \mathrm{M}''_y + \mathrm{M}\alpha^2{}_0 = 0 \text{ , } \mathrm{N}''_x + \mathrm{N}''_y + \mathrm{N}\alpha^2{}_1 = 0 \text{ , et caet. ... ,}$$

in quibus α_0, α_1, ... permanent arbitrariae: quisque videt (a_{40}) expletum iri per singulas

$$z = \mathrm{M}e^{\alpha_0 t} \text{ , } z = \mathrm{N}e^{\alpha_1 t} \text{ , } \ldots .$$

III.° Sit

$$\frac{d^2{}_t z}{dt^2} + k^2 \frac{d_x^4 z}{dx^4} = 0 \ldots (a_{42}).$$

Designantibus M, N, P, ... functiones solius x, pone ut supra

$$z = \mathrm{M}e^{\alpha_0 t} + \mathrm{N}e^{\alpha_1 t} + \mathrm{P}e^{\alpha_2 t} + \ldots (a_{41}) ;$$

habebis

$$\frac{d^2{}_t z}{dt^2} = \mathrm{M}\alpha^2{}_0 e^{\alpha_0 t} + \mathrm{N}\alpha^2{}_1 e^{\alpha_1 t} + \ldots ,$$

$$\frac{d^4{}_x z}{dx^4} = \frac{d^4\mathrm{M}}{dx^4} e^{\alpha_0 t} + \frac{d^4\mathrm{N}}{dx^4} e^{\alpha_1 t} + \ldots ;$$

et adhibitis substitutionibus in (a_{42}),

$$(\mathrm{M}\alpha^2{}_0 + k^2 \frac{d^4\mathrm{M}}{dx^4})e^{\alpha_0 t} + (\mathrm{N}\alpha^2{}_1 + k^2 \frac{d^4\mathrm{N}}{dx^4})e^{\alpha_1 t} + \ldots = 0.$$

Hinc ad M, N, ... determinandas praesto erunt

differentiales aequationes ordinariae

$$M\alpha^2{}_0+k^2\frac{d^4M}{dx^4}=0\,,\quad N\alpha^2{}_1+k^2\frac{d^4N}{dx^4}=0\,,\ \text{et caet.}\dots$$

Notetur illud : sumptis

$$M=\tfrac{1}{2}(M'-M''\sqrt{-1})\,,\ N=\tfrac{1}{2}(N'-N''\sqrt{-1})\,,\dots$$

$$\alpha_0=\alpha'_0\sqrt{-1}\,,\ \alpha_1=\alpha'_1\sqrt{-1}\,,\dots$$

vertetur secundum membrum (a_{43}) in (162. 1.° ex p. 2.^{a})

$$\tfrac{1}{2}[(M'-M''\sqrt{-1})(\cos t\alpha'_0+\sqrt{-1}\,\sin t\alpha'_0)+$$

$$(N'-N''\sqrt{-1})(\cos t\alpha'_1+\sqrt{-1}\,\sin t\alpha'_1)+\dots]\dots(a_{44})\,;$$

sumptis vero

$$M=\tfrac{1}{2}(M'+M''\sqrt{-1})\,,\ N=\tfrac{1}{2}(N'+N''\sqrt{-1})\,,\dots$$

$$\alpha_0=-\alpha'_0\sqrt{-1}\,,\ \alpha_1=-\alpha'_1\sqrt{-1}\,,\dots$$

vertetur membrum illud in

$$\tfrac{1}{2}[(M'+M''\sqrt{-1})(\cos t\alpha'_0-\sqrt{-1}\,\sin t\alpha'_0)+$$

$$(N'+N''\sqrt{-1})(\cos t\alpha'_1+\sqrt{-1}\,\sin t\alpha'_1)+\dots]\dots(a_{45})\,;$$

eritque

$$\begin{aligned}(a_{44})+(a_{45})=&M'\cos t\alpha'_0+N'\cos t\alpha'_1+\dots\\&+M''\sin t\alpha'_0+N''\sin t\alpha'_1+\dots\end{aligned}$$

Ad aequationes nimirum (a_{17}), (a_{40}), (a_{42}), aliasque consimiles lineares integrandas, loco positionis (a_{43}) adhiberi poterit positio

$$\left.\begin{aligned}z=&M'\cos t\alpha'_0+N'\cos t\alpha'_1+P'\cos t\alpha'_2+\dots\\&+M''\sin t\alpha'_0+N''\sin t\alpha'_1+P''\sin t\alpha'_2+\dots\end{aligned}\right\}(a_{46}).$$

CALCULI INTEGRALIS AD GEOMETRIAM APPLICATIO.

DE RECTIFICATIONE CURVARUM.

206. Denotante s arcum curvae planae computatum in ipsa curva a fixo quodam (73) puncto (x_0, y_0) ad mobilem arcus terminum (x, y), cum habeamus (77)

$$ds = \sqrt{\left(1+\frac{dy^2}{dx^2}\right)}dx = \sqrt{\left(1+\frac{dx^2}{dy^2}\right)}dy,$$

erit

$$s = \int_{x_0}^{x} \sqrt{\left(1+\frac{dy^2}{dx^2}\right)}dx = \int_{y_0}^{y} \sqrt{\left(1+\frac{dx^2}{dy^2}\right)}dy \ldots (l).$$

Ad haec: facto (73) angulo $(\tau x) = \theta$, exsistet

$$\sqrt{\left(1+\frac{dy^2}{dx^2}\right)} = \sqrt{(1+tang^2\theta)} = sec\theta,$$

$$\sqrt{\left(1+\frac{dx^2}{dy^2}\right)} = \sqrt{(1+cot^2\theta)} = cosec\theta;$$

proinde

$$s = \int_{x_0}^{x} sec\theta dx = \int_{y_0}^{y} cosec\theta dy \ldots (l').$$

Exempla.

I.° In parabola (87. II.°)

$\frac{dy^2}{dx^2}=\frac{p}{2x}$, et consequenter $s=\int_{x_0}^{x}\sqrt{(\frac{2x+p}{2x})}dx\ldots(l'')$.

Facto (138. II.°) $\frac{2x+p}{2x}=z^2$, prodibit $\sqrt{(\frac{2x+p}{2x})}dx=$

$-p\frac{z^2dz}{(z^2-1)^2}$; et quoniam $\int\frac{z^2dz}{(z^2-1)^2}=\frac{1}{4}\int[\frac{1}{(z+1)^2}-$

$\frac{1}{z+1}+\frac{1}{(z-1)^2}+\frac{1}{z-1}]dz=-\frac{1}{4}[\frac{2z}{z^2-1}-\mathrm{L}(\frac{z-1}{z+1})]+C$,

erit igitur

$$s=\sqrt{(x^2+\frac{px}{2})}-\frac{p}{4}\mathrm{L}\,(\frac{\sqrt{(2x+p)}-\sqrt{2x}}{\sqrt{(2x+p)}+\sqrt{2x}})-$$

$$\sqrt{(x^2_0+\frac{px_0}{2})}+\frac{p}{4}\mathrm{L}\,(\frac{\sqrt{(2x_0+p)}-\sqrt{2x_0}}{\sqrt{(2x_0+p)}+\sqrt{2x_0}});\quad(l''')$$

et computato s a vertice, ubi $x_0=0$,

$$s=\sqrt{(x^2+\frac{px}{2})}-\frac{p}{4}\mathrm{L}\,(\frac{\sqrt{(2x+p)}-\sqrt{2x}}{\sqrt{(2x+p)}+\sqrt{2x}}).$$

II.° In ellipsi (87. II.°)

$$\frac{dy^2}{dx^2}=\frac{b^2x^2}{a^2(a^2-x^2)},\quad\sqrt{(1+\frac{dy^2}{dx^2})}=\sqrt{(\frac{a^4-(a^2-b^2)x^2}{a^2(a^2-x^2)})};$$

in hyperbola (ibid.)

$$\frac{dy^2}{dx^2}=\frac{b^2x^2}{a^2(x^2-a^2)},\quad\sqrt{(1+\frac{dy^2}{dx^2})}=\sqrt{(\frac{(a^2+b^2)x^2-a^4}{a^2(x^2-a^2)})}:$$

et posito (203 ex p. 2.^{a}) in ellipsi

$$\left.\begin{array}{l}a^2-b^2=a^2c^2,\\ \text{in hyperbola}\\ a^2+b^2=a^2c^2,\end{array}\right\}\quad(l^{\text{IV}})$$

erit:

$$\sqrt{(1+\frac{dy^2}{dx})}=\sqrt{(\frac{\pm a^2 \mp c^2x^2}{\pm a^2 \mp x^2})}=\sqrt{(\frac{a^2-c^2x^2}{a^2-x^2})}.$$

Quare in utraque curva

$$s=\int_{x_0}^{x}\sqrt{(\frac{a^2-c^2x^2}{a^2-x^2})}dx \ldots (l^{v}):$$

patet, si ellipseos perimeter vocatur ϖ, fore

$$\varpi=4\int_{0}^{a}\sqrt{(\frac{a^2-c^2x^2}{a^2-x^2})}dx \ldots (l^{vi}).$$

In formula (l^{v}) est $x <$ vel $> a$, prout ipsa (l^{v}) ad ellipsim vel ad hyperbolam pertinet; iccirco poterit in primo casu assumi $x=a\cos\nu$, in secundo $x=\frac{a}{\cos\nu}$: arcus videlicet ellipticus exprimetur quoque per

$$\left.\begin{array}{l} s=-a\int_{\nu_0}^{\nu}\sqrt{(1-c^2\cos^2\nu)}d\nu, \\ \text{hyperbolicus vero per} \\ s=a\int_{\nu_0}^{\nu}\frac{c}{\cos^2\nu}\sqrt{(1-\frac{\cos^2\nu}{c^2})}d\nu. \end{array}\right\}(l^{vii})$$

Ad haec: valoribus $x=0$, $x=a$ respondent in (l^{vi}) $\nu=\frac{\pi}{2}$, $\nu=0$; unde ellipsis perimeter

$$\varpi=-4a\int_{\frac{\pi}{2}}^{0}\sqrt{(1-c^2\cos^2\nu)}d\nu \ldots (l^{viii}).$$

Fiat etiam in (l') prius $x=a\cos\nu$, dein $x=\frac{a}{\cos\nu}$

ut habeamus

$$s=-a\int_{\nu_0}^{\nu} sec\theta sin\nu d\nu\ ,\ s=a\int_{\nu_0}^{\nu} sec\theta\frac{sin\nu}{cos^2\nu}d\nu :$$

ex his respective comparatis cum (l^{VII}) eruitur 1.° quoad ellipsim

$$sec^2\theta\, sin^2\nu=1-c^2cos^2\nu\ ,$$

unde

$$\left.\begin{array}{l} cos^2\nu=\dfrac{sin^2\theta}{1-c^2cos^2\theta}\ ,\ 1-c^2cos^2\nu=\dfrac{1-c^2}{1-c^2cos^2\theta}\ , \\ \nu=arc(cos=\sqrt{[\dfrac{1-c^2}{1-c^2cos^2\theta}]}),\ d\nu=-\dfrac{\sqrt{(1-c^2)}d\theta}{1-c^2cos^2\theta}; \end{array}\right\}(l^{IX})$$

2.° quoad hyperbolam

$$sec^2\theta sin^2\nu=c^2-cos^2\nu\ ,$$

unde

$$\left.\begin{array}{l} cos^2\nu=\dfrac{1-c^2cos^2\theta}{sin^2\theta}\ ,\ \dfrac{c}{cos^2\nu}\sqrt{(1-\dfrac{cos^2\nu}{c^2})}=\dfrac{\sqrt{(c^2-1)}sin\theta}{1-c^2cos^2\theta}\ , \\ \nu=arc(cos=\sqrt{[\dfrac{1-c^2cos^2\theta}{sin^2\theta}]}),d\nu=\dfrac{\sqrt{(c^2-1)}d\theta}{\sqrt{(1-c^2cos^2\theta)}sin\theta} \end{array}\right\}(l^{X})$$

Adhibitis respective substitutionibus ex (l^{IX}), (l^{X}) in (l^{VII}), proveniet

$$s=a(1-c^2)\int_{\theta_0}^{\theta}\frac{d\theta}{(1-c^2cos^2\theta)^{\frac{3}{2}}}\ \ldots\ (l^{XI})$$

pertinens ad utrumque arcum, ellipticum et hyperbolicum : perimeter ellipseos

$$=4a(1-c^2)\int_0^{\frac{\pi}{2}}\frac{d\theta}{(1-c^2\cos^2\theta)^{\frac{3}{2}}}\ldots(l^{XII}).$$

Integralia (l^{V}), (l^{VI}), (l^{VII}), (l^{VIII}), (l^{XI}), (l^{XII}) nequeunt obtineri sub forma finita: restat itaque ut ea per series eliciamus: quod in solis (l^{VII}), (l^{VIII}) perficiemus. Habemus (244 ex p. 1.^{a})

$$\sqrt{(1-c^2\cos^2\nu)}=1-\frac{c^2}{2}\cos^2\nu-\frac{1.c^4}{2.4}\cos^4\nu-$$

$$\frac{1.3c^6}{2.4.6}\cos^6\nu-\frac{1.3.5c^8}{2.4.6.8}\cos^8\nu-\ldots,\quad \frac{c}{\cos^2\nu}\sqrt{\left(1-\frac{\cos^2\nu}{c^2}\right)}=$$

$$\frac{c}{\cos^2\nu}-\frac{1}{2c}-\frac{1.\cos^2\nu}{2.4c^3}-\frac{1.3\cos^4\nu}{2.4.6c^5}-\frac{1.3.5\cos^6\nu}{2.4.6.8c^7}-\ldots;$$

siquidem $c<1$ in prima formula, et $c>1$ in secunda: erit igitur (133) arcus ellipticus

$$\left.\begin{array}{l}
s=a(\nu_0-\nu)+\dfrac{ac^2}{2}\displaystyle\int_{\nu_0}^{\nu}\cos^2\nu\,d\nu+\dfrac{1.ac^4}{2.4}\displaystyle\int_{\nu_0}^{\nu}\cos^4\nu\,d\nu+\\
\qquad\dfrac{1.3ac^6}{2.4.6}\displaystyle\int_{\nu_0}^{\nu}\cos^6\nu\,d\nu+\ldots,\\
\text{hyperbolicus (123)}\\
s=ac(\tang\nu-\tang\nu_0)-\dfrac{a}{2c}(\nu-\nu_0)-\\
\dfrac{1.a}{2.4c^3}\displaystyle\int_{\nu_0}^{\nu}\cos^2\nu\,d\nu-\dfrac{1.3.a}{2.4.6c^5}\displaystyle\int_{\nu_0}^{\nu}\cos^4\nu\,d\nu-\ldots
\end{array}\right\}(l^{XIII}).$$

Integralia $\int_{\nu_0}^{\nu} cos^2\nu d\nu$, $\int_{\nu_0}^{\nu} cos^4\nu d\nu$, $\int_{\nu_0}^{\nu} cos^6\nu d\nu$, ... habentur ex tertia inter formulas jam inventas (143), nimirum

$$\left.\begin{aligned}&\int_{\nu_0}^{\nu} cos^{2n}\nu d\nu = \frac{sin\nu}{2n}\left(cos^{2n-1}\nu + \frac{2n-1}{2n-2}cos^{2n-3}\nu + \ldots\right. \\ &+ \left.\frac{(2n-1)(2n-3)\ldots5.3}{(2n-2)(2n-4)\ldots4.2}cos\nu\right) - \frac{sin\nu_0}{2n}\left(cos^{2n-1}\nu_0 +\right. \\ &\left.\frac{2n-1}{2n-2}cos^{2n-3}\nu_0 + \ldots + \frac{(2n-1)(2n-3)\ldots5.3}{(2n-2)(2n-4)\ldots4.2}cos\,\nu_0\right) \\ &\qquad + \frac{(2n-1)(2n-3)\ldots3.1}{2n(2n-2)(2n-4)\ldots4.2}(\nu-\nu_0)\,;\end{aligned}\right\}(l^{XIV})$$

ex qua cum aperte profluat

$$\int_{\frac{\pi}{2}}^{0} cos^{2n}\nu d\nu = -\frac{(2n-1)(2n-3)\ldots3.1}{2n(2n-2)\ldots4.2}\cdot\frac{\pi}{2}\ldots(l^{XV}),$$

exsurget ellipseos perimeter

$$\left.\begin{aligned}&\varpi = -4a\int_{\frac{\pi}{2}}^{0}\sqrt{(1-c^2cos.^2\nu)}d\nu = \\ &2a\pi\left[1-\left(\frac{1.c}{2}\right)^2 - \frac{1}{3}\left(\frac{1.3c^2}{2.4}\right)^2 - \frac{1}{5}\left(\frac{1.3.5c^3}{2.4.6}\right)^2 - \frac{1}{7}\left(\frac{1.3.5.7c^4}{2.4.6.8}\right)^2 - \ldots\right].\end{aligned}\right\}(l^{XVI})$$

Notetur illud: ex formulis (l^{IX}) ad arcum ellipticum pertinentibus provenit

$$d(c^2cos\nu\,cos\theta = -c^2\,(sin\nu\,cos\theta d\nu + sin\theta cos\nu d\theta) =$$

$$\frac{c^2(1-c^2)cos^2\theta d\theta}{(1-c^2cos^2\theta)^{\frac{3}{2}}}-\frac{c^2(1-cos^2\theta)d\theta}{(1-c^2cos^2\theta)^{\frac{1}{2}}}=$$

$$\frac{1-c^2-(1-c^2cos^2\theta)^2}{(1-c^2cos^2\theta)^2}(1-c^2cos^2\theta)^{\frac{1}{2}}d\theta=$$

$$-(1-c^2cos^2\theta)^{\frac{1}{2}}d\theta+\frac{(1-c^2)d\theta}{(1-c^2cos^2\theta)^{\frac{3}{2}}};$$

unde

$$\left.\begin{aligned}&a(1-c^2)\int_{\theta_0}^{\theta}\frac{d\theta}{(1-c^2cos^2\theta)^{\frac{3}{2}}}=ac^2(cosv\,cos\theta-\\&cosv_0cos\theta_0)+a\int_{\theta_0}^{\theta}\sqrt{(1-c^2cos^2\theta)}d\theta.\end{aligned}\right\}(l^{xviii})$$

exprimit autem

$$a\int_{\theta_0}^{\theta}\sqrt{(1-c^2cos^2\theta)}d\theta$$

arcum ellipticum s', cujus extremitatibus respondent abscissae $x'=acos\theta$, $x'_0=acos\theta_0$; igitur

$$\left.\begin{aligned}&s-s'=ac^2(cosv\,cos\theta-cosv_0cos\theta_0)=\\&ac^2\left(\frac{x}{a}\cdot\frac{x'}{a}-\frac{x_0}{a}\cdot\frac{x'_0}{a}\right)=c^2\frac{xx'-x_0x'_0}{a}\;:\end{aligned}\right\}(l^{xix})$$

sumptis videlicet binis arcubus in ellipsi ita, ut abscissae x et x_0, x' et x'_0 ipsorum extremitatibus respondentes satisfaciant aequationibus

$$\left.\begin{aligned}a^4-a^2(x^2+x'^2)+c^2x^2x'^2=0\,,\\ a^4-a^2(x^2{}_0+x'^2{}_0)+c^2x^2{}_0\,x'^2{}_0=0\,,\end{aligned}\right\}\,(l^{xx})$$

quae manifeste derivantur ex 1.^{a}, vel 2^a (l^{1x}), poterit illorum arcuum differentia analytice exhiberi sub forma finita.

III.° In curva logarithmica (73. II.°) habemus

$$tang\theta=\frac{dy}{dx}=\frac{ba^{\frac{x}{c}}\,L(a)}{c}\,,$$

$$x=\frac{c[L(tang\theta)+L(c)-L(b)-LL(a)]}{L(a)}\,,\,dx=\frac{cd\theta}{L(a)sin\theta cos\theta}\,.$$

Itaque

$$s=\int_{x_0}^{x}sec\theta dx=\frac{c}{L(a)}\int_{\theta_0}^{\theta}\frac{d\theta}{sin\theta cos^2\theta}=$$

$$\frac{c}{L(a)}\int_{\theta_0}^{\theta}\frac{sin^2\theta+cos^2\theta}{sin\theta cos^2\theta}d\theta=\frac{c}{L(a)}\int_{\theta_0}^{\theta}(\frac{tang\theta}{cos\theta}+\frac{1}{sin\theta})d\theta\,;$$

ac proinde

$$s=\frac{c}{L(a)}[sec\theta+L(tang\frac{\theta}{2})-sec\theta_0-L(tang\frac{\theta_0}{2})]\ldots(l^{xxi}).$$

IV.° In cycloide (o^{IV}. 73. I.°) est

$$1+\frac{dx^2}{dy^2}=\frac{2a}{2a-y}\,,\ \text{ideoque}\ s=\int_{y_0}^{y}\sqrt{(\frac{2a}{2a-y})}dy\ldots(l^{xxii}).$$

Fiat (138. II.°)

$$\frac{2a}{2a-y}=z^2\,;$$

prodibit

$$\int \sqrt{\left(\frac{2a}{2a-y}\right)}dy = 4a\int\frac{dz}{z^2} = -\frac{4a}{z} + C = -4a\sqrt{\left(\frac{2a-y}{2a}\right)} + C:$$

igitur

$$s = 4a\sqrt{\left(\frac{2a-y_0}{2a}\right)} - 4a\sqrt{\left(\frac{2a-y}{2a}\right)} \ldots (l^{\text{xxiii}}).$$

Assumpta $y_0=0$, ut arcus computetur ab ipsa cycloidis origine, proveniet

$$\left.\begin{array}{l} s = 4a - 2\sqrt{[2a(2a-y)]}\,; \\ \text{et facta } y=2a, \\ \qquad s=4a. \end{array}\right\} (l^{\text{xxiv}})$$

quod si coordinatarum initium constituatur in cycloidis vertice, erit (73. I.° o''')

$$1 + \frac{dy^2}{dx^2} = \frac{2a}{x};$$

et consequenter

$$\left.\begin{array}{l} s = (2a)^{\frac{1}{2}}\int_{x_0}^{x}\frac{dx}{x^{\frac{1}{2}}} = 2(2a)^{\frac{1}{2}}(x^{\frac{1}{2}} - x_0^{\frac{1}{2}}): \\ \text{posita } x_0=0, \\ \qquad s = 2\sqrt{2ax}\,; \\ \text{et facta } x=2a, \\ \qquad s=4a. \end{array}\right\} (l^{\text{xxv}})$$

Inde colligimus arcum cycloidis, computatum a vertice curvae, duplum esse respondentis chordae in circulo genitore; itemque semicycloidem duplam esse, integramque cycloidem quadruplam diametri ipsius circuli genitoris.

207. Differentiale arcus quoad coordinatas polares z, ω est (77)

$$ds=\sqrt{(dz^2+z^2d\omega^2)}.$$

Jam ut hujus formulae usum unico declaremus exemplo, sit rectificandus arcus ad spiralem logarithmicam (91) spectans. Habemus

$$d\omega=\frac{adz}{z},\ \sqrt{[dz^2+z^2d\omega^2]}=dz\sqrt{[a^2+1]}:$$

proinde

$$s=\sqrt{[a^2+1]}\int_{z_0}^{z}dz=(z-z_0)\sqrt{[a^2+1]}\ldots(l^{\text{xxvi}}).$$

208. Si linea in qua sumitur arcus s, sita est utcumque in spatio, ex dictis (93. 1° : 94) eruentur

$$\left.\begin{aligned}s&=\int_{x_0}^{x}\sqrt{\left(1+\frac{dy^2}{dx^2}+\frac{dz^2}{dx^2}\right)}dx,\\ s&=\int_{x_0}^{x}sec\theta dx.\end{aligned}\right\}(l^{\text{xxvii}})$$

DE QUADRATURA CURVARUM; UBI ET ALIQUID ANNOTATUR CIRCA PROJECTIONES PLANARUM AREARUM IN SUPERFICIEBUS SIMILITER PLANIS

209. Ad obtinendam aream α terminatam ordinatis y_0 et y, intervallo $x-x_0$ sumpto in axe abscissarum ab x_0 ad x, et respondente arcu, praesto est formula (78)

$$\alpha=\int_{x_0}^{x}ydx\ldots(o).$$

Exempla.

I.° In parabola (87. II.°)

$$\left.\begin{array}{l}\alpha=(2p)^{\frac{1}{2}}\int_{x_0}^{x} x^{-\frac{1}{2}}dx=\frac{2}{3}(2p)^{\frac{1}{2}}(x^{\frac{3}{2}}-x_0^{\frac{3}{2}})=\frac{2}{3}(xy-x_0y_0);\\ \text{et computata } \alpha \text{ a vertice curvae,}\\ \alpha=\frac{2}{3}xy.\end{array}\right\}(o')$$

II.° Quoad ellipsim (87. II.°) pone

$$\sqrt{(a^2-x^2)}=zx\ ;$$

quoad hyperbolam,

$$\sqrt{(x^2-a^2)}=zx:$$

erit

$x^2=\frac{\pm a^2}{z^2\pm 1}$, ideoque (125) $\int\sqrt{(\pm a^2\mp x^2)}dx=$

$$\int zxdx=\tfrac{1}{2}x^2z-\tfrac{1}{2}\int x^2dz=\tfrac{1}{2}x\sqrt{(\pm a^2\mp x^2)}\mp\frac{a^2}{2}\int\frac{dz}{z^2\pm 1}.$$

Hinc quoad ellipsim

$$\left.\begin{array}{l}\alpha=\frac{b}{2a}x\sqrt{(a^2-x^2)}+\frac{ab}{2}arc(cot=\frac{\sqrt{(a^2-x^2)}}{x})-\\ \frac{b}{2a}x_0\sqrt{(a^2-x^2_0)}-\frac{ab}{2}arc(cot=\frac{\sqrt{(a^2-x^2_0)}}{x_0})\ ;\end{array}\right\}(o'')$$

quoad hyperbolam

$$\left.\begin{array}{l}\alpha=\frac{b}{2a}x\sqrt{(x^2-a^2)}+\frac{ab}{8}\,\mathrm{L}(\frac{x-\sqrt{(x^2-a^2)}}{x+\sqrt{(x^2-a^2)}})^2-\\ \frac{b}{2a}x_0\sqrt{(x^2_0-a^2)}-\frac{ab}{8}\,\mathrm{L}(\frac{x_0-\sqrt{(x^2_0-a^2)}}{x_0+\sqrt{(x^2_0-a^2)}})^2.\end{array}\right\}(o''')$$

Assumptis in (o'') $x_0=0$, $x=a$, prodibit quarta pars totius areae ellipticae expressa per

$$\frac{ab\pi}{4};$$

unde integra area

$$ab\pi.$$

Ponantur in (o''') $x_0=a$, $b=a$, ut areae computentur a vertice, et hyperbola exsistat aequilatera; erit

$$\alpha=\tfrac{1}{2}x\sqrt{(x^2-a^2)}+\frac{a^2}{8}\,\mathrm{L}\left(\frac{x-\sqrt{(x^2-a^2)}}{x+\sqrt{(x^2-a^2)}}\right)^2 \ldots (o^{IV}).$$

Quod si hyperbola aequilatera ad asymptotos referatur, ejus aequatio erit (201 ex p. 2.^{a})

$$y=\frac{a^2}{2x},\ \text{unde}\ \alpha=\frac{a^2}{4}\,\mathrm{L}\left(\frac{x}{x_0}\right)^2 \ldots (o^{V}).$$

III.° Quoad logarithmicam (173. IV.° ex p. 2.^{a}),

$$\left.\begin{aligned}&\alpha=b\int_{x_0}^{x} a^{\frac{x}{c}}\,dx=\frac{bc}{\mathrm{L}(a)}\left(a^{\frac{x}{c}}-a^{\frac{x_0}{c}}\right);\\ &\text{factis } x=0,\ x_0=-\infty,\\ &\qquad\alpha=\frac{bc}{\mathrm{L}(a)}.\end{aligned}\right\}(o^{VI})$$

IV.° In cycloide (172. III.° ex p. 2.^{a})

$$y=az+a\sin z,\ x=a(1-\cos z);$$

ex quarum secunda

$$dx=a\sin z\,dz.$$

Itaque (125: 143)

$$\left.\begin{array}{l}\Sigma = a^2 \int_{z_0}^{z} (z\sin z + \sin^2 z) dz = a^2[\sin z - \sin z_0 + \\ \frac{1}{2}(z-z_0) + z_0\cos z_0 - z\cos z + \frac{1}{4}(\sin 2z_0 - \sin 2z)]:\end{array}\right\} (o^{VII})$$

assumptis $z_0=0$, $z=\pi$, prodibit semiarea cycloidalis expressa per $\frac{3}{2}a^2\pi$, et integra area per $3a^2\pi$.

210. Permanente abscissarum axe nec non ipsarum origine, praeter curvam cujus ordinatae y concipe et aliam, cujus ordinatae v, ut inquiratur in aream α_1 terminatam ordinatarum differentiis $y-v$, y_0-v_0, et respondentibus curvarum arcubus. Liquet (209) aream α_1 expressum iri per

$$\int_{x_0}^{x} ydx - \int_{x_0}^{x} vdx ;$$

unde (128)

$$\left.\begin{array}{l}\alpha_1 = \int_{x_0}^{x} (y-v)dx , \\ \text{et facto } y-v=t , \\ \alpha_1 = \int_{x_0}^{x} tdx .\end{array}\right\} (o^{VIII})$$

211. Si binae aliae concipiantur curvae, quarum altera habeat ordinatas y', altera ordinatas v', et ponatur $y'-v'=T$, erit respondens area

$$\alpha_2 = \int_{x_0}^{x} Tdx .$$

Quocirca si α_1, α_2 sunt ejusmodi ut ab x_0 ad x_n exsistat constanter

$$\frac{\mathrm{T}}{t}=\frac{a}{b}\ldots(\mathrm{o}^{\mathrm{IX}});$$

denotantibus a et b quantitates constantes, cum in casu prodeat

$$\int_{x_0}^{x_n}\mathrm{T}dx=\frac{a}{b}\int_{x_0}^{x_n}tdx\,,$$

exsistet quoque ab x_0 ad x_n

$$\frac{\alpha_2}{\alpha_1}=\frac{a}{b}\ldots(\mathrm{o}^{\mathrm{X}}).$$

Pone α_2 nihil esse aliud nisi projectionem areae α_1 in plano indefinito P, cujus angulus cum ipsa α_1 designetur per $(\mathrm{P}\alpha_1)$; erit constanter (125 ex p. 2.ª)

$$\frac{\mathrm{T}}{t}=\frac{sos(\mathrm{P}\alpha_1)}{1},\text{ ac proinde }\alpha_2=\alpha_1 cos(\mathrm{P}\alpha_1)\ldots(\mathrm{o}^{\mathrm{XI}}).$$

Formula $(\mathrm{o}^{\mathrm{XI}})$ occasionem nobis praebet aliquid annotandi circa projectiones planarum arearum in superficiebus similiter planis.

1.° sint areae quotvis α', α'', α''', ..., quarum anguli cum planis coordinatis XAY, XAZ, YAZ denotentur per

$$h',\ k',\ i',\quad h'',\ k'',\ i'',\quad h''',\ k''',\ i''',\ \ldots$$

Expriment

$$a'cosh',\ a''cosh'',\ a'''cosh''',\ \ldots$$

projectiones arearum a', a'', a''', ... in plano XAY; rursus

$$a'cosk',\ a''cos\,k'',\ a'''cos\,k''',\ \ldots$$

projectiones in plano XAZ ; demum

$$a'\cos i',\ a''\cos i'',\ a'''\cos i''',\ \ldots$$

projectiones in plano YAZ. Fiant

$$a'\cos h'+a''\cos h''+a'''\cos h'''+\ldots=a,$$

$$a'\cos k'+a''\cos k''+a'''\cos k'''+\ldots=b,$$

$$a'\cos i'+a''\cos i''+a'''\cos i'''+\ldots=c:$$

habemus (177 ex p. 2.^{a})

$$\cos^2 h'+\cos^2 k'+\cos^2 i'=1,\ \cos^2 h''+\cos^2 k''+\cos^2 i''=1,$$

et caet. . . . , itemque

$$\cos(a'a'')=\cos h'\cos h''+\cos k'\cos k''+\cos i'\cos i'',$$

$$\cos(a'a''')=\cos h'h'''+\cos k'\cos k'''+\cos i'\cos i''',$$ et caet..;

quibus positis, haud difficulter pervenietur ad formulam

$$a^2+b^2+c^2=a'^2+a''^2+a'''^2+\ldots+2a'a''\cos(a'a'')+$$
$$2a'a'''\cos(a'a''')+\ldots+2a''a'''\cos(a''a''')+\ldots$$

Et quoniam secundum membrum permanet idem, utcumque collocentur in spatio plana orthogonalia XAY, XAZ, YAZ; idipsum ergo dicendum de primo, idest de summa quadratorum a^2, b^2, c^2.

2.° intelligatur duci planum indefinitum P, quod cum planis XAY, XAZ, YAZ contineat angulos

$$h,\ k,\ i,$$

et in eo quoque fiant arearum a', a'', a''', ... projectiones, quarum summa dicatur μ: erit

$$\mu=a'\cos(\mathrm{P}a')+a''\cos(\mathrm{P}a'')+a'''\cos(\mathrm{P}a''')+\ldots$$

Atqui (177 ex p. 2.^{a})

$$cos(Pa') = cos h\, cos h' + cos k\, cos k' + cos i\, cos i' ,$$

$$cos(Pa'') = cos h\, cos h'' + cos k\, cos k'' + cos i\, cos i'' ,$$

et caet. . . . ; igitur (1.°)

$$\mu = a\, cos h + b\, cos k + c\, cos i.$$

3.° si planum P debeat ejusmodi positionem habere ut summa μ sit maxima omnium quae ad alia plana spectare possunt ; cum inter variabiles $cos\, h$, $cos\, k$, $cos\, i$ vigeat aequatio

$$cos^2 h + cos^2 k + cos^2 i = 1 ,$$

erunt igitur (61 : exempl.)

$$cos h = \frac{a}{\sqrt{(a^2+b^2+c^2)}} , \; cos k = \frac{b}{\sqrt{(a^2+b^2+c^2)}} ,$$

$$cos i = \frac{c}{\sqrt{(a^2+b^2+c^2)}} ,$$

et maxima

$$\mu = \sqrt{(a^2+b^2+c^2)} .$$

4.° concipiatur alterum planum indefinitum P_1, quod cum planis XAY, XAZ, YAZ contineat angulos h_1, k_1, i_1 : factis etiam in P_1 projectionibus arearum a', a'', a''', . . . , et ejusmodi projectionum summa designata per μ_1, erit (2.°)

$$\mu_1 = a\, cos h_1 + b\, cos k_1 + c\, cos i_1 .$$

Jam si plano P respondet maxima μ, exsistet (3.°)

$$cos(PP_1) = cos h\, cos h_1 + cos k\, cos k_1 + cos i\, cos i_1 = \frac{a\, cos h_1 + b\, cos k_1 + c\, cos i_1}{\sqrt{(a^2+b^2+c^2)}} ;$$

igitur

$$\mu_1 = \sqrt{(a^2+b^2+c^2)}\, cos(PP_1) :$$

summa videlicet projectionum manebit eadem in omnibus planis P_1 aeque inclinatis ad P. Accedente insuper (PP_1) ad angulum rectum, decrescet μ_1 usque adeo, donec, facto $(PP_1)=90°$, evadat demum $\mu_1=0$.

212. Si curva refertur ad coordinatas polares z, ω; substitutis in ultima formula (79) valoribus x, y, dx, dy (75: item 174 ex p. 2.^{a}), proveniet $d\alpha'=-\frac{1}{2}z^2d\omega$, unde

$$\alpha'=-\frac{1}{2}\int_{\omega_0}^{\omega} z^2 d\omega \ldots (0^{XI}).$$

DE QUADRATURA SUPERFICIERUM CURVARUM

213. Aequatio ad superficiem curvam sit

$$z=f(x\,,y)\ldots(a);$$

in ipsa vero superficie considerentur bina puncta, alterum fixum $(x_0\,,y_0\,,z_0)$, alterum mobile $(x\,,y\,,z)$: transeant per $(x_0\,,y_0\,,z_0)$ duo plana, alterum perpendiculare axi AX, alterum axi AY; itemque per $(x\,,y\,,z)$ alia duo plana pariter perpendicularia, alterum axi AX, alterum axi AY. Ista quatuor plana secabunt superficiem (a) secundum quatuor curvas, planumque XAY secundum quatuor rectas: pars superficiei quatuor illis terminata curvis, utpote functio variabilium independentium x, y, designetur per $\varphi(x\,,y)$; area quatuor illis terminata rectis vocetur U, quae cum sit aperte rectangularis, erit

$$U=(x-x_0)(y-y_0)\ldots(a'):$$

rectangulum U est projectio areae $\varphi(x\,,y)$ in plano XAY. Ad haec: si x, y recipiunt incrementa infinitesima Δx, Δy, exprimunt

$$\Delta_x\varphi(x\,,y)\,,\ \Delta_x U=(y-y_0)\Delta x\ \ldots\ (a'')$$

incrementa arearum $\varphi(x\,,y)$ et U respondentia soli Δx, et

$$\Delta_y\Delta_x\varphi(x\,,y)\,,\ \Delta_y\Delta_x U=\Delta y.\Delta x\ \ldots\ (a''')$$

incrementa areolarum (a'') quoad Δy. Ducatur nunc planum tangens superficiem (a) in puncto $(x\,,y\,,z)$, et sub angulo θ constitutum cum plano XAY; intelligantur insuper per punctum $(x+\Delta x\,,y+\Delta y\,,z+\Delta z)$ transire bina plana, alterum perpendiculare axi AX, alterum axi AY; demum haec duo plana, nec non bina illa quae jam posuimus transire per punctum $(x\,,y\,,z)$ concipiantur produci donec secent planum tangens. Si quadrilaterum ex quatuor hujusmodi sectionibus exsurgens in plano tangente dicatur ν, cum secunda ex areolis (a''') sit projectio ipsius ν in plano XAY, erit (211. o^{XI})

$$\nu=\frac{\Delta y.\Delta x}{cos\theta}\,.$$

Ex dictis autem (76: 77) intelligimus fore

$$lim.\frac{\Delta_y\Delta_x\varphi(x\,,y)}{\nu}=1.$$

itaque (39 : 40 : 42)

$$\frac{1}{cos\theta}=lim.\frac{\Delta_y\Delta_x\varphi(x,y)}{\Delta y.\Delta x}=lim.\frac{\dfrac{\Delta_y\left(\dfrac{\Delta_x\varphi(x,y)}{\beta}\right)}{\beta}}{\dfrac{\Delta y}{\beta}\cdot\dfrac{\Delta x}{\beta}}=\frac{d_yd_x\varphi(x,y)}{dy\ dx}\,;$$

seu

$$\frac{d_yd_x\varphi(x\,,y)}{dy\ dx}=sec\theta\ \ldots\ (a^{\text{IV}}).$$

Ex (a^{IV}), integrando quoad y, eruimus

$$\frac{d_x(\varphi(x\,,y)-\varphi(x\,,y_0))}{dx}=\int_{y_0}^{y} sec\theta dy\;;$$

iterumque integrando quoad x,

$$\varphi(x,y)-\varphi(x,y_0)-\varphi(x_0,y)+\varphi(x_0,y_0)=\int_{x_0}^{x}\int_{y_0}^{y} sec\theta dy dx.$$

Sed evanescente projectione (a') evanescit quoque area $\varphi(x\,,y)$, ideoque

$$\varphi(x\,,y_0)=0\,,\;\varphi(x_0\,,y)=0\,,\;\varphi(x_0\,,y_0)=0.$$

Igitur

$$\varphi(x\,,y)=\int_{x_0}^{x}\int_{y_0}^{y} sec\theta dy dx\,,$$

seu, ob ultimam $(h'''.\ 109)$,

$$\varphi(x\,,y)=\int_{x_0}^{x}\int_{y_0}^{y}\sqrt{[1+(\frac{dz}{dx})^2+(\frac{dz}{dy})^2]}\,dy dx. \quad (a^{\text{V}})$$

214. Quoniam (213)

$$lim.\frac{\Delta_y\Delta_x\varphi(x\,,y)}{\Delta y.\Delta x}=\frac{1}{cos\theta},$$

iccirco

$$\frac{\Delta_y\Delta_x\varphi(x\,,y)}{\Delta y.\Delta x}=sec\theta+\omega_x+\omega_y\,;$$

denotant ω_x, ω_y bina infinitesima, alterum vergens ad $lim.=0$ quando transitur ad limites quoad x, alterum vergens ad $lim.=0$ quando fit gradus ad limites quoad y. Est autem

$$\frac{\Delta_y \Delta_x \varphi(x, y)}{\Delta y . \Delta x} = \frac{\dfrac{\Delta_y (\Delta_x \varphi(x, y))}{\beta}}{\dfrac{\Delta y}{\beta} . \Delta x} :$$

itaque facto ad limites gradu quoad y, prodibit:

$$\frac{d_y \Delta_x \varphi(x, y)}{dy . \Delta x} = sec\theta + \omega_x ;$$

et consequenter prima ex areolis (a'')

$$\Delta_x \varphi(x, y) = \Delta x . \int_{y_0}^{y} (sec\theta + \omega_x) dy \ldots (a^{VI}).$$

215. Superficies (a) secetur nunc 1.° duobus planis perpendicularibus axi AX, quorum alteri respondeat abscissa fixa x_0, alteri abscissa variabilis x; 2.° binis superficiebus cylindraceis, quarum generatrices (188 ex p. 2.^{a}) sint parallelae axi AZ, et quarum aequationes exhibeantur per

$$y = F(x), \; y = F_1(x) \ldots (a^{VII}),$$

exsistente $F_1(x) > F(x)$: quatuor inde orientur sectiones curvilineae in superficie (a), et pars superficiei istis quatuor sectionibus terminata, utpote functio abscissae x, poterit exprimi per $\psi(x)$; jam in aream $\psi(x)$ inquiramus. Ducantur bina plana perpendicularia axi AY, alterum per extremitatem ordinatae $F(x)$, alterum per extremitatem ordinatae $F_1(x+\Delta x)$; ducatur quoque per extremitatem abscissae $x+\Delta x$ aliud planum perpendiculare axi AX: haec tria plana una cum illo e superioribus, cui posuimus respondere abscissam x, secabunt superficiem (a), et areola quatuor ejusmodi sectionibus terminata exprimetur (75: a^{VI}) per

$$\Delta x \,.\int_{F(x)}^{F_1(x+\Delta x)} (sec\theta+\omega_x)dy.$$

Hac vero comparata cum areola $\Delta\psi(x)$, habebimus

$$lim. \frac{\Delta\psi(x)}{\Delta x \,.\displaystyle\int_{F(x)}^{F_1(x+\Delta x)} (sec\theta+\omega_x)dy} = 1\ ;$$

igitur

$$d\psi(x) = \int_{F(x)}^{F_1(x)} sec\theta dy dx\ ,\ \psi(x) = \int_{x_0}^{x} \int_{F(x)}^{F_1(x)} sec\theta dy dx \ldots (a^{VIII}).$$

216. E secunda (a^{VIII}) facile transitur ad quadraturam superficiei, quae gignitur rotatione curvae planae circa axem AX. Concipiatur curva generans in plano XAY, ponaturque ejus aequatio exprimi per

$$u = F_1(x) \ldots (a^{IX}),$$

ut sit (180. II.° ex p. 2.^{a})

$$y^2+z^2 = F^2{}_1(x) \ldots (a^X)$$

aequatio ad genitam superficiem; unde

$$\frac{dz}{dx} = \frac{F_1(x) \,.\, F'_1(x)}{(F^2{}_1(x)-y^2)^{\frac{1}{2}}}\ ,\ \frac{dz}{dy} = -\frac{y}{(F^2{}_1(x)-y^2)^{\frac{1}{2}}}\ ,$$

et consequenter ob ultimam (h'''. 109)

$$sec\theta = \frac{\sqrt{(1+F'^2{}_1(x))}}{\sqrt{(F^2{}_1(x)-y^2)}} F_1(x) \ldots (a^{XI}).$$

Ad haec: in secunda (a^{VII}) fiat $F(x) = 0$; pars su-

perficiei (a^{x}) comprehensa inter plana coordinata XAY, XAZ et inter alia duo plana perpendicularia axi AX, quorum alteri respondet abscissa x_0, alteri abscissa x, poterit exhiberi per

$$\psi_1(x) = \int_{x_0}^{x}\int_{0}^{F_1(x)} \sec\theta dy\, dx =$$

$$\int_{x_0}^{x}\int_{0}^{F_1(x)} \frac{\sqrt{(1+F'^2{}_1(x))}}{\sqrt{(F^2{}_1(x)-y^2)}} F_1(x) dy dx =$$

$$\int_{x_0}^{x} \sqrt{(1+F'^2{}_1(x))} F_1(x) dx \int_{0}^{F_1(x)} \frac{dy}{\sqrt{(F^2{}_1(x)-y^2)}}.$$

Est autem (138. III.°)

$$\int_{0}^{F_1(x)} \frac{dy}{\sqrt{(F^2{}_1(x)-y^2)}} = 2arc(cot=1) - 2arc(cot=\infty) = \frac{\pi}{2};$$

praeterea denotante ν normalem curvae (a^{IX}), habemus (73)

$$\sqrt{[1+F'^2{}_1(x)]}.F_1(x) = \nu \ldots (a^{\text{XII}}):$$

igitur

$$\psi_1(x) = \frac{\pi}{2}\int_{x_0}^{x} \nu dx \ldots (a^{\text{XIII}})$$

217. Ut usum formulae (a^{XIII}) declaremus exemplo, proponatur aequatio

$$y^2+z^2=\frac{r^2}{p^2}(p^2-x^2)$$

ad superficiem (221 ex p. 2.^{a}) solidi generati rotatione ellipseos

$$y^2=\frac{r^2}{p^2}(p^2-x^2)$$

circa axem $2p$. Sit 1.° $p>r$: facto

$$p^2-r^2=c^2p^2 \ldots (a^{XIV}),$$

eruetur ex (a^{XII})

$$v=\frac{r}{p}\sqrt{(p^2-c^2x^2)}=\frac{c}{p}\sqrt{(\frac{p^2}{c^2}-x^2)} \ldots (a^{XV});$$

unde (209. II.°)

$$\left.\begin{aligned}\psi_1(x)=\pi\frac{cr}{2p}\int_{x_0}^{x}\sqrt{(\frac{p^2}{c^2}-x^2)}dx=\pi\frac{cr}{4p}[x\sqrt{(\frac{p^2}{c^2}-x^2)}+\\ \frac{p^2}{c^2}arc(cot=\frac{\sqrt{(\frac{p^2}{c^2}-x^2)}}{x})-x_0\sqrt{(\frac{p^2}{c^2}-x^2_0)}-\\ \frac{p^2}{c^2}arc(cot=\frac{\sqrt{(\frac{p^2}{c^2}-x^2_0)}}{x_0})\,].\end{aligned}\right\}(a^{XVI})$$

Sit 2.° $p<r$: facto

$$r^2-p^2=c^2r^2 \ldots (a^{XVII}),$$

erit

$$v=\frac{r}{p}\sqrt{(p^2+\frac{c^2r^2}{p^2}x^2)}=\frac{cr^2}{p^2}\sqrt{(\frac{p^4}{c^2r^2}+x^2)} \ldots (a^{XVIII});$$

ideoque (209. II.°)

$$\psi_1(x)=\pi\frac{cr^2}{2p^2}\int_{x_0}^{x}\sqrt{(\frac{p^4}{c^2r^2}+x^2)}dx=\pi\frac{cr^2}{4p^2}[x\sqrt{(\frac{p^4}{c^2r^2}+$$

$$x^2) - \frac{p^4}{4c^2r^2}\mathrm{L}\left(\frac{x-\sqrt{(\frac{p^4}{c^2r^2}+x^2)}}{x+\sqrt{(\frac{p^4}{c^2r^2}+x^2)}}\right)^2 - x_0\sqrt{(\frac{p^4}{c^2r^2}+x^2{}_0)}+$$

$$\frac{p^4}{4c^2r^2}\mathrm{L}\left(\frac{x_0-\sqrt{(\frac{p^4}{c^2r^2}+x^2{}_0)}}{x_0+\sqrt{(\frac{p^4}{c^2r^2}+x^2{}_0)}}\right)^2] \ldots\ldots (a^{\text{XIX}}).$$

Animadvertendo quod ex (a^{XIV}) provenit

$$\sqrt{(\frac{p^2}{c^2}-p^2)}=\frac{r}{c},$$

et ex (a^{XVII})

$$\sqrt{(\frac{p^4}{c^2r^2}+p^2)}=\frac{p}{c},$$

si tota solidi superficies vocetur S, eruemus in primo casu

$$\left.\begin{array}{l} S=8\pi\frac{cr}{2p}\int_0^p\sqrt{(\frac{p^2}{c^2}-x^2)}dx=2\pi r^2+2\pi\frac{pr}{c}arc(\cot=\frac{r}{cp}), \\ \text{in secundo} \\ S=8\pi\frac{cr^2}{2p^2}\int_0^p\sqrt{(\frac{p^4}{c^2r^2}+x^2)}dx=2\pi r^2+\pi\frac{p^2}{c}\mathrm{L}\frac{1+c}{1-c}. \end{array}\right\} (a^{\text{XX}})$$

DE SOLIDORUM CUBATURA.

218. Resumpta altera e binis superficiebus cylindraceis $(a^{\text{VII}}$ 215), inveniendum sit solidum ν terminatum ipsa superficie usque ad altitudinem datam

$=a$, et basi $\alpha=\int_{x_0}^{x} ydx$ computata in plano XAY ab x_0 ad x.

Abscissa x evadat $x+\Delta x$; incrementum Δv, rectumque parallelepipedum $ay\Delta x$ erunt evidenter ejusmodi, ut exsistat

$$lim.\frac{\Delta v}{ay\Delta x}=1:$$

hinc

$$dv=aydx, \text{ et } v=a\int_{x_0}^{x} ydx=a\alpha \ldots (k).$$

219. Solidum v sit nunc terminatum utcumque: per extremitatem abscissae x ducatur planum perpendiculare axi AX, quod secet solidum v; et hujusmodi intersectionis area designetur per $f(x)$: super basi $f(x)$ intelligatur constitutus cylindrus rectus habens altitudinem $=\Delta x$, qui proinde (218 : k) exprimetur per $f(x)\Delta x$. Liquet fore

$$lim.\frac{\Delta v}{f(x)\Delta x}=1;$$

igitur

$$dv=f(x)dx, \text{ et } v=\int_{x_0}^{x} f(x)dx \ldots (k').$$

Quod ad aream planam $f(x)$ spectat, ea definietur ex jam traditis de quadratura curvarum. Proponatur v. gr. invenienda soliditas ellipsoidis (221 ex p. 2.ª)

$$\frac{z^2}{r^2}+\frac{y^2}{q^2}+\frac{x^2}{p^2}=1.$$

Area $f(x)$ est elliptica (ibid.); ad quam spectat ae-

quatio

$$\frac{z^2}{r^2}+\frac{y^2}{q^2}=1-\frac{x^2}{p^2}:$$

et consequenter semiaxes

$$r\sqrt{(1-\frac{x^2}{p^2})},\ q\sqrt{(1-\frac{x^2}{p^2})}.$$

Hinc (209 : II.°)

$$f(x)=\pi rq(1-\frac{x^2}{p^2});$$

et

$$v=\pi rq\int_{x_0}^{x}(1-\frac{x^2}{p^2})dx=\pi rq(x-\frac{x^3}{3p^2}-x_0+\frac{x_0^3}{3p^2})\ldots(k'').$$

Jam si fiant $x=p$, $x_0=-p$, prodibit soliditas totius ellipsoidis

$$V=\frac{4\pi}{3}pqr\ldots(k''').$$

220. Ponatur solidum terminari et superficiebus curvis

$$z_0=\chi_0(x,y),\ z_1=\chi_1(x,y)\ldots(k^{\text{IV}}),$$

et superficiebus cylindraceis

$$y_0=\mathrm{f}_0(x),\ y_1=\mathrm{f}_1(x)\ldots(k^{\text{V}}),$$

et superficiebus planis

$$x_0=c_0,\ x_1=c_1\ldots(k^{\text{VI}}):$$

erit (210)

$$f(x)=\int_{y_0}^{y_1}(z_1-z_0)dy=\int_{y_0}^{y_1}\int_{z_0}^{z_1}dzdy;$$

unde (219)

$$v=\int_{x_0}^{x_1}\int_{y_0}^{y_1}(z_1-z_0)dydx=\int_{x_0}^{x_1}\int_{y_0}^{y_1}\int_{z_0}^{z_1}dzdydx\ldots(k^{VII}).$$

221. Si et aliud proponitur solidum v_1 terminatum superficiebus curvis

$$Z_0=\psi_0(x,y),\ Z_1=\psi_1(x,y)\ldots(k^{VIII}),$$

superficiebus insuper cylindraceis (k^{V}), et superficiebus planis (k^{VI}), habebimus

$$v_1=\int_{x_0}^{x_1}\int_{y_0}^{y_1}(Z_1-Z_0)dydx.$$

Quare si inter rectas Z_1-Z_0, z_1-z_0 parallelas axi AZ, quarum altera superficiebus (k^{VIII}) intercipitur, altera superficiebus (k^{IV}), constans servetur ratio $\frac{a}{b}$, cum prodeat

$$Z_1-Z_0=\frac{a}{b}(z_1-z_0),$$

erit quoque

$$\frac{v_1}{v}=\frac{a}{b}\ldots(k^{IX}).$$

222. In plano XAY intelligatur describi curva

$$y=F(x)\ldots(k^{X}),$$

ex cujus revolutione circa axem AX gignatur solidum v: intersectio $f(x)$ erit manifeste circularis, cujus radius $=y$; unde

$$f(x)=\pi y^2,\ \text{et}\ v=\pi\int_{x_0}^{x}y^2dx\ldots(k^{XI}).$$

Sic v. gr. in plano XAY descripta cycloide, cujus

vertex in A, basis vero secetur bifariam et ad angulos rectos ab axe AX, erit (209 : IV.°)

$$y^2dx=a^3(z^2\sin z+2z\sin^2 z+\sin^3 z)dz;$$

unde

$$v=\pi a^3\int_{z_0}^{z}(z^2\sin z+2z\sin^2 z+\sin^3 z)dz.$$

Sunt autem (143 : 147 : item 162. 1.° ex p. 2.ª)

$$\int z^2\sin z dz=2z\sin z-(z^2-2)\cos z+C,$$

$$2\int z\sin^2 z dz=\int z(1-\cos 2z)dz=\frac{z^2}{2}-\frac{z}{2}\sin 2z-\frac{1}{4}\cos 2z+C,$$

$$\int \sin^3 z dz=-\frac{1}{3}(\sin^2 z+2)\cos z+C:$$

igitur

$$\left.\begin{aligned}&v=\pi a^3[\frac{z^2}{2}+z(2\sin z-\tfrac{1}{2}\sin 2z-z\cos z)+\\&\frac{\cos z}{3}(4-\sin^2 z)-\frac{1}{4}\cos 2z-\frac{z^2_0}{2}-z_0(2\sin z_0-\\&\tfrac{1}{2}\sin 2z_0-z_0\cos z_0)-\frac{\cos z_0}{3}(4-\sin^2 z_0)+\frac{1}{4}\cos^2 2z_0]\end{aligned}\right\}(k^{XII}).$$

Factis $z_0=0$, $z=\pi$, proveniet solidum genitum ex revolutione cycloidis circa suum axem,

$$V=\pi a^3(\frac{3\pi^2}{2}-\frac{8}{3})\ldots(k^{XIII}).$$

Quod si debeat inveniri solidum genitum ex revolutione cycloidis circa suam basim, posita cycloidis origine in A, intelligenda erit ejus basis constituta juxta axem AX. Habemus autem in casu (73. I.° item 172. III.° ex p. 2.ª)

$$y=a(1+\cos z)\ ,\ x=a(\pi-z-\sin z)\ ,$$

et consequenter

$$y^2dx=-a^3(1+\cos z)^3dz=-8a^3\cos^6\frac{z}{2}dz\ :$$

itaque (143)

$$v=-8a^3\pi\int_{z_0}^{z}\cos^6\frac{z}{2}dz=-8a^3\pi\left[\frac{\sin\frac{z}{2}}{3}\left(\cos^5\frac{z}{2}+\frac{5}{4}\cos^3\frac{z}{2}+\frac{5.3}{4.2}\cos\frac{z}{2}\right)-\frac{\sin\frac{z_0}{2}}{3}\left(\cos^5\frac{z_0}{2}+\frac{5}{4}\cos^3\frac{z_0}{2}+\frac{5.3}{4.2}\cos\frac{z_0}{2}\right)+\frac{5}{8}\left(\frac{z}{2}-\frac{z_0}{2}\right)\right]\ \ldots\ (k^{xiv}).$$

Assumptis $z_0=\pi$, $z=-\pi$, prodibit solidum genitum ex revolutione cycloidis circa suam basim

$$V=5a^3\pi^2\ \ldots\ (k^{xv}).$$

223. Praeter curvam (k^x) describatur in plano XAY alia curva

$$Y=F_1(x)\ \ldots\ (k^{xvi})\ ,$$

ut ambae revolvantur circa axem AX : solidum v interceptum superficiebus genitis rotatione binarum (k^x), (k^{xvi}) manifeste exprimetur per

$$\pi\int_{x_0}^{x}Y^2dx-\pi\int_{x_0}^{x}y^2dx.$$

Hinc

$$v=\pi\int_{x_0}^{x}(Y^2-y^2)dx\ \ldots\ (k^{xvii}).$$

224. Bina solida

$$\nu_1=\int_{x_0}^{x_n} f_1(x)dx\ ,\ \nu_2=\int_{x_0}^{x_n} f_2(x)dx\ \ldots\ (k^{\mathrm{xviii}})$$

ponantur esse ejusmodi, ut inter sectionum areas $f_1(x)$, $f_2(x)$ constans servetur ratio $\frac{a}{b}$. Erit

$$f_1(x)=\frac{a}{b}f_2(x)\ ,\ \text{ideoque} \int_{x_0}^{x} f_1(x)dx=\frac{a}{b}\int_{x_0}^{x} f_2(x)dx;$$

seu

$$\frac{\nu_1}{\nu_2}=\frac{a}{b}\ \ldots\ (k^{\mathrm{xix}}).$$

DUO PROPONUNTUR PROBLEMATA, QUORUM ALTERUM RESPICIT TRAJECTORIAS CURVARUM, ALTERUM SOLUTIONES PARTICULARES.

225. Ad 1.^{um} quod spectat, detur aequatio

$$y=f(x\ ,\ C)\ \ldots\ (h),$$

ut inveniatur curva

$$y=\psi(x)\ \ldots\ (h')$$

secans sub dato angulo curvas omnes, quas praebet (h) quum, permanentibus caeteris, constanti arbitrariae C varii tribuuntur valores: curva (h') dicitur *trajectoria* curvarum (h).

Ex puncto $(x\ ,y)$, in quo curva (h') secat curvam (h), duc tangentes t, τ, alteram ad (h), alteram ad (h'), sitque a tangens trigonometrica dati anguli $(t\tau)$: quoniam (73)

$$tang(tx)=f'\ ,\ tang(\tau x)=\psi'\ ,$$

ac praeterea (35. 8.° ex p. 2ª.)

$$(tt) = (\tau x) - (tx);$$

icciroo (127. 7.° ex p. 2ª)

$$a = \frac{\psi' - f'}{1 + f'\psi'} \ldots (h''').$$

Jam si ex (h) et (h'') eliminetur C, prodibit differentialis aequatio complectens omnia intersectionum puncta; cujus integratio suppeditabit quaesitam curvam (h'). Ponatur $f(x, \mathrm{C}) = \mathrm{C}x$, ut (h) recidat in $y = \mathrm{C}x$: erit $a = \dfrac{\mathrm{C} - \psi'}{1 + \mathrm{C}\psi'}$; et eliminata C,

$$a = \frac{\psi' - \dfrac{y}{x}}{1 + \dfrac{y}{x}\psi'}, \text{ seu } a = \frac{\dfrac{dy}{dx} - \dfrac{y}{x}}{1 + \dfrac{y}{x}\dfrac{dy}{dx}};$$

hinc

$$a(xdx + ydy) = xdy - ydx, \text{ seu}$$

$$\frac{a(xdx + ydy)}{x^2 + y^2} + \frac{ydx - xdy}{x^2 + y^2} = 0;$$

ideoque (151)

$$a\mathrm{L}(x^2 + y^2)^{\frac{1}{2}} + arc\left(tang = \frac{x}{y}\right) = \mathrm{C}_1.$$

Est autem

$$arc\left(tang = \frac{x}{y}\right) = \frac{\pi}{2} - arc\left(tang = \frac{y}{x}\right);$$

mutata igitur constanti arbitraria $\mathrm{C}_1 - \dfrac{\pi}{2}$ in $a\mathrm{L}(\mathrm{C}_2)$, emerget

$$aL\frac{(x^2+y^2)^{\frac{1}{2}}}{C_2}=arc(tang=\frac{y}{x});$$

aequatio (91: exempl.) ad spiralem logarithmicam.

Quod ad 2.^{um} spectat, proponatur invenienda ejusmodi curva, ut, si ad ejus tangentes ab origine coordinatarum ducuntur perpendicula, haec exsistant semper aequalia inter se.

Aequationum (73. o″)

$$u-y=\frac{dy}{dx}(v-x), u=-\frac{dx}{dy}v$$

altera pertinet ad rectas tangentes curvam, altera ad respondentia perpendicula; iisque resolutis quoad u et v, inde prodibunt coordinatae punctorum, in quibus perpendicula occurrunt tangentibus, nimirum

$$v=\frac{(xdy-ydx)dy}{dx^2+dy^2}, u=-\frac{(xdy-ydx)dx}{dx^2+dy^2}:$$

denotante igitur a communem perpendiculorum longitudinem, erit

$$\sqrt{[v^2+u^2]}=\frac{xdy-ydx}{\sqrt{[dx^2+dy^2]}}=a;$$

unde differentialis ad quaesitam curvam aequatio

$$xdy-ydx=a\sqrt{[dx^2+dy^2]}\ldots(k);$$

quae cum admittat solutiones particulares jam determinatas (199. IV°.), profecto suppeditabit circulum

$$y^2+x^2-a^2=0,$$

curvamque praeterea $\quad\Bigg\}\,(k')$

$$y^2(a^2-x^2)-(x^2+a^2)^2=0.$$

Ad haec: quoad (k) habemus (199. IV.°) $y''=0$, seu

$\frac{d^2y}{dx^2}=0$; proinde $dy=C_1dx$, $y=C_1x+C_2$: et adhibitis substitutionibus in (k), emerget $C_2=-a\sqrt{[1+C^2_1]}$; completum videlicet integrale ipsius (k) erit

$$y=C_1x-a\sqrt{[1+C^2_1]},$$

repraesentabitque rectas omnes quibus tanguntur curvae (k').

Generatim, si per $P=0$ exhibeatur particularis solutio, et per $Q=0$ integrale completum differentialis aequationis (o. 153), repraesentabit $Q=0$ curvas omnes tangentes curvam $P=0$: patet (73) ex eo quod (Q) recidat in

$$\frac{dy}{dx}=-\frac{M}{N}.$$

DE METHODO MAXIMORUM ET MINIMORUM AD INVENIENDAS FUNCTIONES INCOGNITAS APPLICATA.

226. Sit functio incognita $y=F(x)$; et compendii causa exhibeantur per y', y'', y''', ... derivatae $F'(x)$, $F''(x)$, $F'''(x)$, ...; debeat vero definiri y ita, ut integrale

$$\int_{x_0}^{x_n} f(x, y, y', y'', y''', \ldots)dx \;\ldots (b')$$

exsistat maximum minimumve. Quisque videt quaestionem, si ad res geometricas traducatur, eo recidere ut proponatur invenienda linea in qua ipsum (b') evadat maximum minimumve.

Si maximo minimove integralis (b') valori respondet functio $F(x)$, certe in viciniis maximi minimive valoris poterit respondens functio generatim reprae-

sentari per

$$F(x)+\omega\varphi(x) \text{ seu } y+\omega\varphi(x);$$

denotat ω quantitatem infinitesimam, φ functionem indeterminatam. Exprimat Δ differentiam inter maximum minimumve integralis (b') valorem et valorem illum, quem recipit idem (b') quum pro y adhibetur $y+\omega\varphi$; erit (128)

$$\Delta=\int_{x_0}^{x_n}[f(x,y+\omega\varphi,y'+\omega\varphi',y''+\omega\varphi'',..)-f(x,y,y',y'',..)]dx;$$

quae differentia, utcumque caeteroquin sumitur infinitesima ω, retineat oportet idem signum, si quidem (b') debet esse maximum minimumve. Ad haec: prodibit (b') maximum si permanet $\Delta<0$, minimum si permanet $\Delta>0$. Quoniam differentiae Δ idem signum insit oportet, quantitas igitur

$$\frac{\Delta}{\omega}$$

erit ejusmodi, ut illius signum necessario mutetur mutato signo denominatoris ω. Est autem (63)

$$\frac{\Delta}{\omega}=\int_{x_0}^{x_n}[\varphi(x)\frac{df(x,y,y',y'',...)}{dy}+\varphi'(x)\frac{df(x,y,y',...)}{dy'}+$$

$$\varphi''(x)\frac{df(x,y,y',...)}{dy''}+...]dx+$$

$$\frac{\omega}{2}\int_{x_0}^{x_n}[\varphi^2(x)\frac{d^2f(x,y,y',...)}{dy^2}+\varphi'^2(x)\frac{d^2f(x,y,y',...)}{dy'^2}+...$$

$$+2\varphi(x)\varphi'(x)\frac{d^2f(x,y,y',...)}{dydy'}+...+\omega_1]dx;$$

designat ω_1 quantitatem infinitesimam: expressis ita-

quae compendii causa per

$$R\,,\ S\,,\ T\,,\ V\,,\ \ldots$$

quantitatibus

$$\frac{df(x,y,y',..)}{dy},\ \frac{df(x,y,y',..)}{dy'},\ \frac{df(x,y,y',..)}{dy''},\ \frac{df(x,y,y',..)}{dy'''},..$$

restat ut in casu maximi minimive exsistat

$$\int_{x_0}^{x_n}[\varphi(x)R+\varphi'(x)S+\varphi''(x)T+\varphi'''(x)V+\ldots]dx=0.$$

Jamvero (132)

$$\int_{x_0}^{x_n}S\varphi'(x)dx=S_{x_n}\varphi(x_n)-S_{x_0}\varphi(x_0)-\int_{x_0}^{x_n}\varphi(x)\frac{dS}{dx}dx\ ,$$

$$\int_{x_0}^{x_n}T\varphi''(x)dx=T_{x_n}\varphi'(x_n)-T_{x_0}\varphi'(x_0)-\int_{x_0}^{x_n}\varphi'(x)\frac{dT}{dx}dx=$$

$$T_{x_n}\varphi'(x_n)-T_{x_0}\varphi'(x_0)-T'_{x_n}\varphi(x_n)+T'_{x_0}\varphi(x_0)+$$

$$\int_{x_0}^{x_n}\varphi(x)\frac{d^2T}{dx^2}dx\,,\ \int_{x_0}^{x_n}V\varphi'''(x)dx=V_{x_n}\varphi''(x)-$$

$$V_{x_0}\varphi''(x_0)-V'_{x_n}\varphi'(x_n)+V'_{x_0}\varphi'(x_0)+V''_{x_n}\varphi(x_n)-$$

$$V''_{x_0}\varphi(x_0)-\int_{x_0}^{x_n}\varphi(x)\frac{d^3V}{dx^3}dx\,,\ \text{et caet.}\ldots;$$

quare, adhibitis substitutionibus,

$$\left.\begin{array}{l}(S_{x_n}-T'_{x_n}+V''_{x_n}-\ldots)\varphi(x_n)-(S_{x_0}-T'_{x_0}+\\ V''_{x_0}-\ldots)\varphi(x_0)+(T_{x_n}-V'_{x_n}+\ldots)\varphi'(x_n)-\\ (T_{x_0}-V'_{x_0}+\ldots)\varphi'(x_0)+(V_{x_n}-\ldots)\varphi''(x_n)-\\ (V_{x_0}-\ldots)\varphi''(x_0)+\ldots+\int_{x_0}^{x_n}(R-\dfrac{dS}{dx}+\\ \dfrac{d^2T}{dx^2}-\dfrac{d^3V}{dx^3}+\ldots)\varphi(x)dx.\end{array}\right\}=0\ldots(b'')$$

Aequatio haec debet valere, utcumque se habet $\varphi(x)$; et cum fieri possit ut exsistant

$$\varphi(x_n)=0\ ,\ \varphi'(x_n)=0\ ,\ \varphi''(x_n)=0\ ,\ \ldots$$
$$\varphi(x_0)=0\ ,\ \varphi'(x_0)=0\ ,\ \varphi''(x_0)=0\ ,\ \ldots$$

quin tamen $\varphi(x)$ desinat esse indeterminata, cumque in ejusmodi casu redigatur (b'') ad

$$\int_{x_0}^{x_n}(R-\frac{dS}{dx}+\frac{d^2T}{dx^2}-\frac{d^3V}{dx^3}+\ldots)\varphi(x)dx=0\ldots(b'''),$$

consequens est ut ipsa (b'') importet binas aequationes distinctas; nimirum (b'''), ex qua (136. 1.°)

$$R-\frac{dS}{dx}+\frac{d^2T}{dx^2}-\frac{d^3V}{dx^3}+\ldots=0\ldots(b^{\text{IV}}),$$

et

$$\left.\begin{array}{l}(S_{x_n}-T'_{x_n}+V''_{x_n}-\ldots)\varphi(x_n)-(S_{x_0}-T'_{x_0}+\\ V''_{x_0}-\ldots)\varphi(x_0)+(T_{x_n}-V'_{x_n}+\ldots)\varphi'(x_n)-\\ (T_{x_0}-V'_{x_0}+\ldots)\varphi'(x_0)+(V_{x_n}-\ldots)\varphi''(x_n)-\\ \qquad (V_{x_0}-\ldots)\varphi''(x_0)+\ldots\end{array}\right\}=0\ldots(b^{\text{V}})$$

Ex (b^{IV}) integrata habebitur quaesita relatio inter y

et x, seu talis functio $y=\mathrm{F}(x)$ ut, substitutis y, y', y'', ... in (b'), inde prodeat maximus minimusve valor ipsius (b'); modo tamen constantes arbitrariae ita determinentur, ut expleatur etiam (b^{v}).

227. Si desunt integrali (b') functiones derivatae y', y'', ..., ut habeatur

$$\int_{x_0}^{x_n} f(x, y)dx \ldots (b^{\mathrm{vi}}),$$

binae (b^{iv}), (b^{v}) manifeste redigentur ad unicam $\mathrm{R}=0$, seu

$$\frac{df(x, y)}{dy}=0 \ldots (b^{\mathrm{vii}}),$$

cujus ope invenietur functio $y=\mathrm{F}(x)$ praebens maximum minimumve (b^{vi}).

228. Expositam methodum declarabunt quae sequuntur

Exempla.

I.° Detur

$$\int_{x_0}^{x_n} (ax-y^2)ydx:$$

emerget (227 : b^{vii}) inter y, x relatio

$$ax-3y^2=0;$$

unde

$$\int_{x_0}^{x_n} (ax-y^2)ydx=\frac{4a}{3.5}(x_n^{\frac{5}{2}}-x_0^{\frac{5}{2}})\sqrt{\frac{a}{3}}.$$

Est autem (226)

$$\Delta=\int_{x_0}^{x_n}[(ax-(y+\omega\varphi(x))^2)(y+\omega\varphi(x))-(ax-y^2)y],$$

$$-\omega^2\int_{x_0}^{x_n}(3\varphi^2(x)\sqrt{\frac{ax}{3}}+\omega\varphi^3(x))dx,$$

videlicet negativa; ponimus namque $x_n > x_0$: functio igitur $y=\sqrt{\frac{ax}{3}}$ suppeditabit maximum integralis propositi valorem. Sequitur lineam, in qua integrale illud fit maximum, fore parabolam cujus parameter $=\frac{a}{3}$.

II.° Detur

$$\int_{x_0}^{x_n}(2xy-y^2)dx:$$

habebimus (227 : b^{VII}) relationem

$$y=x$$

spectantem ad lineam rectam, quae transit per coordinatarum originem, et abscissarum axem secat sub angulo $=45°$; hinc

$$\int_{x_0}^{x_n}(2xy-y^2)dx=\frac{x^3_n-x^3_0}{3}.$$

Est autem

$$\Delta=\int_{x_0}^{x_n}[2x(x+\omega\varphi(x))-(x+\omega\varphi(x))^2-2x^2+x^2]dx=-\omega^2\int_{x_0}^{x_n}\varphi^2(x)dx:$$

itaque relatio illa suppeditat maximum.

III.° Inter lineas, quarum extrema puncta ad eas-

dem abscissas x_0, x_n, easdemque ordinatas y_0, y_n referuntur, invenire maximam minimamve, seu eam in qua (206)

$$\int_{x_0}^{x_n} \sqrt{(1+y'^2)}dx$$

sit maximum vel minimum: etsi constat solas rectas lineas solutioni inservire, proderit nihilominus hoc exemplum ad traditam (226) methodum declarandam. Habemus

$$R=\frac{d\sqrt{(1+y'^2)}}{dy}=0,\ S=\frac{d\sqrt{(1+y'^2)}}{dy'}=\frac{y'}{\sqrt{(1+y'^2)}},T=0,V=0,$$

et caet.....; igitur (b^{IV}) evadet

$$\frac{dS}{dx}=0,$$

unde

$$S=C,\ \text{seu}\ \frac{y'}{\sqrt{(1+y'^2)}}=C,\ \text{ideoque}\ y'=\frac{C}{\sqrt{1-C^2}}=C_1,\ \frac{dy}{dx}=C_1;$$

et

$$y=C_1x+C_2.$$

ad rectam pertinens lineam: hinc vero

$$\int_{x_0}^{x_n} \sqrt{(1+y'^2)}dx=(x_n-x_0)\sqrt{(1+C_1^2)}\ \ldots (b^{\text{VIII}}).$$

Quoniam linearum extremitates ponuntur habere easdem coordinatas, iccirco abscissis x_0, x_n respondebunt ordinatae y_0, y_n haud variabiles quum ab una linea transitur ad alteram, et consequenter

$$\varphi(x_n)=0\,,\ \varphi(x_0)=0\,,\ \varphi'(x_n)=0\,,\ \varphi'(x_0)=0\,,$$
$$\varphi''(x_n)=0\,,\ \varphi''(x_0)=0\,,\ \text{et caet.}\ \ldots;$$

non pluribus opus est ut expleatur (b^{v}). Designantes nunc per Ω quantitatem infinitesimam, habemus

$$\Delta=\int_{x_0}^{x_n}[\sqrt{(1+(C_1+\omega\varphi'(x))^2)}-\sqrt{(1+C^2{}_1)}]dx=$$

$$\int_{x_0}^{x_n}[\frac{2C_1\omega\varphi'(x)+\omega^2\varphi'^2(x)}{2(1+C^2{}_1)^{\frac{1}{2}}}-\frac{C^2{}_1\omega^2\varphi'^2(x)}{2(1+C^2{}_1)^{\frac{3}{2}}}+\omega^2\Omega]dx=$$

$$\int_{x_0}^{x_n}[\frac{C_1\omega\varphi'(x)}{(1+C^2{}_1)^{\frac{1}{2}}}+\frac{\omega^2\varphi'^2(x)}{(1+C^2{}_1)^{\frac{3}{2}}}+\omega^2\Omega]dx=$$

$$\frac{C_1\omega}{(1+C^2{}_1)^{\frac{1}{2}}}(\varphi(x_n)-\varphi(x_0))+\omega^2\int_{x_0}^{x_n}[\frac{\varphi'^2(x)}{2(1+C^2{}_1)^{\frac{3}{2}}}+\Omega]dx=$$

$$\omega^2\int_{x_0}^{x_n}[\frac{\varphi'^2(x)}{2(1+C^2{}_1)^{\frac{3}{2}}}+\Omega]dx:$$

quia igitur prodit Δ positiva, certe non maximum sed minimum dumtaxat obtinebit.

Si linearum extremitates non habent easdem coordinatas, cum ob $T=0$, $V=0$, et caet. . . . aequatio (b^{v}) redigatur ad

$$S_{x_n}\varphi(x_n)-S_{x_0}\varphi(x_0)=0\,,$$

cumque in ea qua sumus hypothesi neque sint $\varphi(x_n)=0$, $\varphi(x_0)=0$, neque ob x_n, x_0 datas possit universim assumi $S_{x_n}\varphi(x_n)=S_{x_0}\varphi(x_0)$, restat igitur ut exsistant

$S_{x_n}=0$, $S_{x_0}=0$, seu $y'x_n=0$, $y'x_0=0$; quod eo redit in praesenti quaestione ut evanescente tangente trigonometrica anguli, quem linea recta efficit cum abscissarum axe, linea illa fiat ipsi axi parallela.

Notetur illud: si x_n, x_0 habentur pro abscissis curvarum.

$$y_n=\psi(x_n),\ y_0=\chi(x_0) \ldots (g),$$

profecto quae rectae ab una ad alteram curvam duci possunt, licet earum extremitates ad easdem abscissas minime referantur, adhuc tamen habebunt longitudinem $=(b^{\text{VIII}})$; nisi quod valor tangentis trigonometricae C_1 exhibebitur per

$$C_1=\frac{y_n-y_0}{x_n-x_0}=\frac{\psi(x_n)-\chi(x_0)}{x_n-x_0}:$$

quare ad ejusmodi rectarum minimam spectabunt (56)

$$\frac{d(b^{\text{VIII}})}{dx_n}=0,\ \frac{d(b^{\text{VIII}})}{dx_0}=0;$$

id est

$$\left.\begin{aligned} x_n-x_0+[\psi(x_n)-\chi(x_0)]\psi'(x_n)=0, \\ x_n-x_0+[\psi(x_n)-\chi(x_0)]\chi'(x_0)=0: \end{aligned}\right\}(g')$$

et quoniam

$$\psi(x_n)-\chi(x_0)=(x_n-x_0)C_1;$$

iccirco

$$1+C_1\psi'(x_n)=0,\ 1+C_1\chi'(x_0)=0.$$

Ex quibus colligimus (73: item 172. I.° 4.° ex p. 2.ª) brevissimam distantiam inter duas curvas fore lineam rectam ipsis curvis normalem in punctis, quorum coordinatae x_n, y_n, x_0, y_0 eruuntur ex (g) et (g').

IV.° Inter lineas, quarum extremitates iisdem abscissis x_0, x_n, iisdemque ordinatis y_0, y_n sunt

praeditae, eam invenire in qua

$$\int_{x_0}^{x_n} \sqrt{\left(\frac{1+y'^2}{x}\right)} dx$$

sit maximum minimumve. Habemus

$$R=0, \; S=\frac{d\sqrt{\left(\frac{1+y'^2}{x}\right)}}{dy'}=\frac{y'}{\sqrt{x}\sqrt{(1+y'^2)}}, T=0, V=0,$$

et caet. . . . ; propterea (b^{IV} 226)

$$\frac{dS}{dx}=0,$$

unde

$$S=C, \text{ idest } \frac{y'}{\sqrt{x}\sqrt{[1+y'^2]}}=C; \text{ et } y'^2=\frac{xC^2}{1-C^2x} \ldots (b^{\text{IX}}).$$

Ex (b^{IX})

$$1+y'^2=\frac{1}{1-C^2x} \text{ ideoque } (138.\text{III.}^\circ) \int_{x_0}^{x_n} \sqrt{\left(\frac{1+y'^2}{x}\right)} dx=$$

$$\int_{x_0}^{x_n} \frac{dx}{\sqrt{(x-C^2x^2)}}=\frac{2}{C}\left[arc\left(cot=\frac{\sqrt{(1-C^2x_n)}}{C\sqrt{x_n}}\right)-\right.$$

$$\left. arc\left(cot=\frac{\sqrt{(1-C^2x_0)}}{C\sqrt{x_0}}\right)\right].$$

Ipsa (b^{IX}) est ad cycloidem; nam expressa C^2 per $\frac{1}{2a}$, et adhibita $2a-x$ pro x, vertetur (b^{IX}) in

$$\frac{dy^2}{dx^2}=\frac{2a-x}{x},$$

quam in ordine ad cycloidem jam invenimus (73. I.°). Quod ad (b^{v}) pertinet, ea manifeste impletur; omnium enim linearum extremitates ponuntur easdem habere ordinatas y_0, y_n, ac proinde

$$\varphi(x_0)=0\ ,\ \varphi(x_n)=0\ ,\ldots$$

Ad haec: denotante Ω quantitatem infinitesimam, prodit

$$\Delta=\int_{x_0}^{x_n}\frac{1}{\sqrt{x}}[(1+(y'+\omega\varphi'(x))^2)^{\frac{1}{2}}-(1+y'^2)^{\frac{1}{2}}]dx=$$

$$\int_{x_0}^{x_n}[\frac{y'\omega\varphi'(x)}{\sqrt{x}\sqrt{(1+y'^2)}}+\frac{\omega^2\varphi'^2(x)}{2\sqrt{x}(1+y'^2)^{\frac{3}{2}}}+\omega^2\Omega]dx=$$

$$\int_{x_0}^{x_n}[C\omega\varphi'(x)+\frac{\omega^2\varphi'^2(x)}{2\sqrt{x}(1+y'^2)^{\frac{3}{2}}}+\omega^2\Omega]dx=$$

$$C\omega[\varphi(x_n)-\varphi(x_0)]+\omega^2\int_{x_0}^{x_n}[\frac{\varphi'^2(x)}{2\sqrt{x}(1+y'^2)^{\frac{3}{2}}}+\Omega]dx=$$

$$\omega^2\int_{x_0}^{x_n}[\frac{\varphi'^2(x)}{2\sqrt{x}(1+y'^2)^{\frac{3}{2}}}+\Omega]dx\ ,$$

videlicet positiva; ideoque, excluso maximo, obtinebit minimum. Si eae dumtaxat linearum extremitates quibus respondet abscissa x_0 ponuntur habere communem ordinatam y_0, erunt equidem $\varphi(x_0)=0$, $\varphi'(x_0)=0$, ...; at non item $\varphi(x_n)=0$, $\varphi'(x_n)=0$, ...; quare cum ob $T=0$, $V=0$, ... redigatur (b^{v}) ad

$$S_{x_n}\varphi(x_n)=0\ ,$$

sumenda erit $S_{x_n}=0$; ideoque $y_{x_n}=0$. Cyclois nimirum, in qua

$$\int_{x_0}^{x_n} \sqrt{\left(\frac{1+y'^2}{x}\right)}dx$$

est minimum, debet ejusmodi positionem habere ut per punctum ad quod pertinet x_n ducta tangente, haec exsistat parallela axi abscissarum.

229. Aliquando proponitur invenienda curva praedita maximi minimive proprietate inter eas tantum, quibus una, duae, pluresve proprietates communes sunt: sic v. gr. inter solas curvas, in quibus integrale

$$\int_{x_0}^{x_n} f_1(x, y, y', \ldots)dx$$

retinet eumdem valorem, quaeri potest curva illa in qua

$$\int_{x_0}^{x_n} f_2(x, y, y', \ldots)dx$$

est maximum minimumve. Haec nova quaestio facile traducitur ad hactenus pertractatam de invenienda ejusmodi curva inter caeteras omnes: assumpta constanti et arbitraria α, pone

$$\int_{x_0}^{x_n} f_1(x, y, y', \ldots)dx = \varkappa, \text{ unde } \alpha\int_{x_0}^{x_n} f_1(x, y, y' \ldots)dx = \varkappa'$$

et facto

$$\alpha\int_{x_0}^{x_n} f_1(x,y,y',\ldots)dx + \int_{x_0}^{x_n} f_2(x,y,y',\ldots)dx = \int_{x_0}^{x_n} f(x,y,y',\ldots)dx,$$

certe quae relatio inter x et y ex caeteris omnibus

relationibus praebet maximum minimumve

$$\int_{x_0}^{x_n} f(x, y, y', \ldots)dx,$$

eadem ipsa manifeste suppeditabit

$$\int_{x_0}^{x_n} f_2(x, y, y', \ldots)dx$$

maximum minimumve quoad eas relationes, in quibus

$$\int_{x_0}^{x_n} f_1(x, y, y', \ldots)dx$$

retinet eumdem valorem.

Exempla.

I.° Inter omnes curvas ejusdem longitudinis jungentes puncta fixa (x_0, y_0), (x_n, y_n) invenire, eam quae maximam vel minimam aream continet. Erunt (206 : 209)

$$\int_{x_0}^{x_n} f_1(x, y, y', \ldots)dx = \int_{x_0}^{x_n} \sqrt{(1+y'^2)}dx,$$

$$\int_{x_0}^{x_n} f_2(x, y, y', \ldots)dx = \int_{x_0}^{x_n} y dx;$$

unde

$$\int_{x_0}^{x_n} f(x, y, y', \ldots)dx = \int_{x_0}^{x_n} [y + \alpha\sqrt{(1+y'^2)}]dx; \text{ et } (226)$$

$$R=1, \; S=\alpha\frac{y'}{\sqrt{(1+y'^2)}}, \; T=0, \; V=0, \ldots.$$

Aequatio (b^{IV}) fiet

$$1-\alpha\frac{1}{dx}d\frac{y'}{\sqrt{(1+y'^2)}}=0\text{ , seu } dx-\alpha d\frac{y'}{\sqrt{(1+y'^2)}}=0\text{ ;}$$

hinc

$$x-\alpha\frac{y'}{\sqrt{(1+y'^2)}}=\mathrm{C},\ y'=\frac{x-\mathrm{C}}{\sqrt{[\alpha^2-(x-\mathrm{C})^2]}}\text{ ,}$$

$$dy=\frac{(x-\mathrm{C})dx}{\sqrt{[\alpha^2-(x-\mathrm{C})^2]}},\ y=-\sqrt{[\alpha^2-(x-\mathrm{C})^2]}+\mathrm{C}_1\text{ ; et}$$

$$(y-\mathrm{C}_1)^2+(x-\mathrm{C})^2=\alpha^2\text{ ,}$$

aequatio ad circulum. Quod ad (b^{V}) pertinet, ea manifeste impletur; omnium enim curvarum extremitates habent ex hypothesi easdem coordinatas. Restat videndum utrum Δ sit positiva, vel negativa: habemus

$$\Delta=\int_{x_0}^{x_n}[y+\omega\varphi(x)+\alpha\sqrt{(1+(y'+\omega\varphi'(x))^2)}-y-\alpha\sqrt{(1+y'^2)}]dx=$$

$$\int_{x_0}^{x_n}[\omega\varphi(x)+\alpha(\frac{y'\omega\varphi'(x)}{\sqrt{(1+y'^2)}}+\frac{\omega^2\varphi'^2(x)}{2(1+y'^2)^{\frac{5}{2}}}+\omega^2\Omega)]dx=$$

$$\int_{x_0}^{x_n}[\omega\varphi(x)+(x-\mathrm{C})\omega\varphi'(x)+\alpha(\frac{\omega^2\varphi'^2(x)}{2(1+y'^2)^{\frac{5}{2}}}+\omega^2\Omega)]dx.$$

Est autem

$$\int_{x_0}^{x_n}(\omega\varphi(x)+(x-\mathrm{C})\omega\varphi'(x))dx=$$

$$\omega(x_n\varphi(x_n)-x_0\varphi(x_0)-\mathrm{C}\varphi(x_n)+\mathrm{C}\varphi(x_0))=0\text{ :}$$

itaque

$$\Delta = \alpha\omega^2 \int_{x_0}^{x_n} \left[\frac{\varphi'^2(x)}{2(1+y'^2)^{\frac{3}{2}}} + \Omega \right] dx\,;$$

quae, cum sit positiva, denotat minimum. At notandum quod e superiori valore y' prodit functio derivata secundi ordinis positiva, nimirum

$$y'' = \frac{\alpha^2}{(\alpha^2-(x-C)^2)^{\frac{3}{2}}}\,;$$

ideoque (80) arcus circuli intra datos terminos constitutus convexitatem obvertet axi abscissarum: si concavitatem obverteret, profecto maximam contineret aream.

II.° Inter omnes curvas ejusdem longitudinis determinare illam, quae sese revolvendo circa axem abscissarum x gignit solidum maximae minimaeve superficiei: curvarum extremitates ad easdem referimus coordinatas x_0, y_0, x_n, y_n.

Erunt (206 : 216)

$$\int_{x_0}^{x_n} f_1(x, y, y' \ldots) dx = \int_{x_0}^{x_n} \sqrt{(1+y'^2)}\, dx\,,$$

$$\int_{x_0}^{x_n} f_2(x, y, y', \ldots) dx = \int_{x_0}^{x_n} y\sqrt{(1+y'^2)}\, dx\,,$$

unde

$$\int_{x_0}^{x_n} f(x, y, y' \ldots) dx = \int_{x_0}^{x_n} [(\alpha+y)\sqrt{(1+y'^2)}]\, dx\,;$$

et (226)

$$R = \sqrt{(1+y'^2)}\,,\ S = (\alpha+y)\frac{y'}{\sqrt{(1+y'^2)}},\ T=0,\ V=0,\ \ldots$$

Quare (226 : b^{IV})

$$\sqrt{(1+y'^2)} - \frac{y'^2}{\sqrt{(1+y'^2)}} - \frac{1}{dx}(\alpha+y)d\frac{y'}{\sqrt{(1+y'^2)}} = 0\,,$$

seu

$$\frac{1}{\sqrt{(1+y'^2)}} - \frac{1}{dx}(\alpha+y)\frac{dy'}{(1+y'^2)^{\frac{3}{2}}} = 0\,;$$

hinc

$$(1+y'^2)dx = (\alpha+y)dy' \ldots (b^{\text{X}}):$$

est autem $dx = \dfrac{dy}{y'}$; igitur

$$\frac{dy}{\alpha+y} = \frac{y'dy'}{1+y'^2}:$$

et integrando,

$L(\alpha+y) = L[C\sqrt{(1+y'^2)}]$, ideoque $\alpha+y = C\sqrt{(1+y'^2)} \ldots (b^{\text{XI}})$.

Substituatur valor $\alpha+y$ ex (b^{XI}) in (b^{X}); proveniet

$$dx = \frac{Cdy'}{\sqrt{(1+y'^2)}}\,,$$

ex cujus integratione (138. III.°)

$$x - C_1 = CL[y' + \sqrt{(1+y'^2)}].$$

Aequatio haec praebet

$$e^{\frac{x-C_1}{C}} = y' + \sqrt{(1+y'^2)}\,,\ \text{unde}\ y' = \frac{e^{\frac{x-C_1}{C}} - e^{-\frac{x-C_1}{C}}}{2}\,;$$

et adhibita substitutione in (b^{XI}), factisque $\alpha+y=u$, $x - C_1 = v$,

$$u=C\frac{e^{\frac{\nu}{c}}+e^{-\frac{\nu}{c}}}{2},$$

aequatio ad curvam, quae vulgo dicitur *catenaria.*

III.° Si in curvis praeter longitudinem permanet etiam area, quaeritur curva illa, quae sui revolutione circa axem abscissarum gignit solidum maximum minimumve.

Erit (206 : 209 : 222)

$$\int_{x_0}^{x_n} f(x, y, y', \ldots)dx=\int_{x_0}^{x_n}[\alpha\sqrt{(1+y'^2)}+\alpha_1 y+y^2]dx,$$

unde

$$R=\alpha_1+2y, \quad S=\frac{\alpha y'}{\sqrt{(1+y'^2)}};$$

et (b^{IV} : 226)

$$\alpha_1+2y-\alpha\frac{1}{dx}d\frac{y'}{\sqrt{(1+y'^2)}}=0, \text{ seu } \alpha_1+2y-\alpha\frac{1}{dx}\frac{dy'}{(1+y'^2)^{\frac{3}{2}}}=0.$$

Hinc ob $dy=y'dx$,

$$(\alpha_1+2y)dy-\alpha\frac{y'dy'}{(1+y'^2)^{\frac{3}{2}}}=0;$$

ex cujus integratione

$$(\alpha_1+y)y+\frac{\alpha}{\sqrt{(1+y'^2)}}+C=0:$$

inde autem

$$y'=\frac{\sqrt{[\alpha^2-(C+\alpha_1 y+y^2)^2]}}{C+\alpha_1 y+y^2},$$

seu

$$dx = \frac{(C+\alpha_1 y+y^2)dy}{\sqrt{[\alpha^2-(C+\alpha_1 y+y^2)^2]}},$$

differentialis aequatio primi ordinis ad curvam, quae dicitur *elastica*.

230. Sint binae functiones incognitae

$$y=F(x),\ z=F_1(x)$$

determinandae ita, ut

$$\int_{x_0}^{x_n} f(x,y,y',z',y'',z'',\ldots)dx \ldots (b^{XII})$$

exsistat maximum minimumve.

Valor z ponatur jam cognitus, jamque substitutus in (b^{XII}) una cum derivatis z', z'', ...: in ea qua sumus hypothesi variabit (b^{XII}) quoad solam y, et consequenter prodibunt quoad y (226) aequationes (b^{IV}), (b^{V}), nisi quod litterae

$$R,\ S,\ T,\ V,\ldots$$

hic denotant quantitates

$$\frac{df(x,y,z,y',z',y'',z'',\ldots)}{dy},\ \frac{df(x,y,z,y',z',\ldots)}{dy'},$$

$$\frac{df(x,y,z,y',z',\ldots)}{dy''},\ldots$$

At quemadmodum habita est z ut nota et y ut incognita, sic poterat haberi y ut nota et z ut incognita: hinc denotantibus

$$r,\ s,\ t,\ v,\ldots$$

quantitates

$$\frac{df(x,y,z,y',z',\ldots)}{dz}\,,\ \frac{df(x,y,z,y',z',\ldots)}{dz'}\,,$$

$$\frac{df(x,y,z,y',z',\ldots)}{dz''}\,,\ \frac{df(x,y,z,y',z',\ldots)}{dz'''}\,,\ldots$$

exsurgent quoad z

$$r-\frac{ds}{dx}+\frac{d^2t}{dx^2}-\frac{d^3v}{dx^3}+\ldots=0\ldots(b^{XIII}),$$

$$\left.\begin{array}{l}(sx_n-t'x_n+v''x_n-\ldots)\varphi_1(x_n)-(sx_0-t'x_0+\\ v''x_0-\ldots)\varphi_1(x_0)+(tx_n-v'x_n+\ldots)\varphi'_1(x_n)-\\ (tx_0-v'x_0+\ldots)\varphi'_1(x_0)+(vx_n-\ldots)\varphi''_1(x_n)-\\ \quad(vx_0-\ldots)\varphi''_1(x_0)+\ldots\end{array}\right\}=0\ldots(b^{XIV})$$

Exempla.

I.° Inter lineas jungentes bina puncta fixa utcumque in spatio collocata invenire maximam minimamve, seu eam, in qua (208)

$$\int_{x_0}^{x_n}\sqrt{(1+y'^2+z'^2)}dx$$

existit maximum minimumve. Erunt

$$R=0\,,\ S=\frac{y'}{\sqrt{(1+y'^2+z'^2)}}\,,\ T=0\,,\ V=0\ldots,$$

$$r=0\,,\ s=\frac{z'}{\sqrt{(1+y'^2+z'^2)}}\,,\ t=0\,,\ v=0\,,\ldots$$

et binae (b^{IV}), (b^{XIII}) evadent

$$\frac{dS}{dx}=0\,,\ \frac{ds}{dx}=0\,;$$

igitur

$$S=\frac{y'}{\sqrt{(1+y'^2+z'^2)}}=const., \ s=\frac{z'}{\sqrt{(1+y'^2+z'^2)}}=const.;$$

ex quibus profluunt

$$y'=C\,,\ z'=C_2 \ \text{seu}\ \frac{dy}{dx}=C\,,\ \frac{dz}{dx}=C_2\,;$$

ideoque

$$y=Cx+C_1\,,\ z=C_2x+C_3 \ldots (b^{xv})\,,$$

aequationes ad lineam rectam. Quod ad (b'), (b^{xiv}) spectat, eae manifeste implentur; cum ex hypothesi extremitates linearum omnium gaudeant iisdem coordinatis. Demum

$$\Delta=\int_{x_0}^{x_n}[\sqrt{(1+(y'+\omega\varphi'(x))^2+(z'+\omega\varphi'_1(x))^2)}-$$

$$\sqrt{(1+y'^2+z'^2)}]dx=\int_{x_0}^{x_n}[\frac{C\omega\varphi'(x)+C_2\omega\varphi'_1(x)}{(1+C^2+C^2_2)^{\frac{1}{2}}}+$$

$$\frac{\omega^2(\varphi'^2(x)+\varphi'^2_1(x)+(C_2\varphi'(x)-C\varphi'_1(x))^2)}{2(1+C^2+C^2_2)^{\frac{3}{2}}}+\omega^2\Omega]dx:$$

est autem

$$\int_{x_0}^{x_n}[\frac{C\omega\varphi'(x)+C_2\omega\varphi'_1(x)}{(1+C^2+C^2_2)^{\frac{1}{2}}}]dx=$$

$$\frac{\omega}{(1+C^2+C^2_2)^{\frac{1}{2}}}[C(\varphi(x_n)-\varphi(x_0))+C_2(\varphi_1(x_n)-\varphi_1(x_0))]=0;$$

prodit igitur Δ positiva; ideoque, excluso maximo, obtinet minimum.

Juvat hic notare illud: longitudo rectae (b^{xv}), cujus extrema puncta definiuntur coordinatis x_0, y_0, z_0, x_n, y_n, z_n, exprimitur per

$$\left.\begin{array}{l}\int_{x_0}^{x_n} V(1+y'^2+z'^2)dx = \int_{x_0}^{x_n} V(1+C^2+C^2_2)dx = \\ \qquad (x_n - x_0)V(1+C^2+C^2_2)\end{array}\right\}(b^{xvi}).$$

Fac, ut coordinatae illae pertineant ad binas superficies curvas

$$z_0 = \chi(x_0, y_0), \; z_n = \psi(x_n, y_n) \ldots (h);$$

habebis

$$C_2 = \frac{z_n - z_0}{x_n - x_0} = \frac{\psi(x_n, y_n) - \chi(x_0, y_0)}{x_n - x_0}, \; C = \frac{y_n - y_0}{x_n - x_0}.$$

Ad minimam rectarum omnium, quae ab una ad alteram superficiem duci possunt, spectabunt (56)

$$\frac{d(b^{xvi})}{dx_0} = 0, \quad \frac{d(b^{xvi})}{dy_0} = 0, \quad \frac{d(b^{xvi})}{dx_n} = 0, \quad \frac{d(b^{xvi})}{dy_n} = 0,$$

seu

$$\left.\begin{array}{l} x_n - x_0 + [\psi(x_n, y_n) - \chi(x_0, y_0)]\dfrac{d\chi(x_0, y_0)}{dx_0} = 0, \\ y_n - y_0 + [\psi(x_n, y_n) - \chi(x_0, y_0)]\dfrac{d\chi(x_0, y_0)}{dy_0} = 0, \end{array}\right\}(h')$$

$$\left.\begin{array}{l} x_n - x_0 + [\psi(x_n, y_n) - \chi(x_0, y_0)]\dfrac{d\psi(x_n, y_n)}{dx_n} = 0, \\ y_n - y_0 + [\psi(x_n, y_n) - \chi(x_0, y_0)]\dfrac{d\psi(x_n, y_n)}{dy_n} = 0, \end{array}\right\}(h'')$$

et quoniam

$$\psi(x_n, y_n)-\chi(x_0, y_0)=(x_n-x_0)C_1, y_n-y_0=(x_n-x_0)C,$$

iccirco

$$1+C_2\left(\frac{dz_0}{dx_0}\right)=0, \quad C+C_2\left(\frac{dz_0}{dy_0}\right)=0,$$

$$1+C_2\left(\frac{dz_n}{dx_n}\right)=0, \quad C+C_2\left(\frac{dz_n}{dy_n}\right)=0:$$

harum prima et secunda praebent

$$\frac{1}{C_2}=-\frac{dz_0}{dx_0}, \quad \frac{C}{C_2}=-\frac{dz_0}{dy_0},$$

tertia, et quarta

$$\frac{1}{C_2}=-\frac{dz_n}{dx_n}, \quad \frac{C}{C_2}=-\frac{dz_n}{dy_n}.$$

Jam vero

$$\frac{1}{C_2}, \quad \frac{C}{C_2}$$

denotant tangentes angulorum, quos projectiones lineae rectae minimae in planis XAZ, YAZ continent cum axe AZ: itaque expressis per X, Y, Z angulis, quos recta illa minima efficit cum axibus AX, AY, AZ, factisque

$$\sqrt{\left[1+\left(\frac{dz_0}{dx_0}\right)^2+\left(\frac{dz_0}{dy_0}\right)^2\right]}=k_0, \quad \sqrt{\left[1+\left(\frac{dz_n}{dx_n}\right)^2+\left(\frac{dz_n}{dy_n}\right)^2\right]}=k_n,$$

exsurgent (184 ex p. 2.a)

$$\cos X=-\frac{1}{k_0}\frac{dz_0}{dx_0}, \quad \cos Y=-\frac{1}{k_0}\frac{dz_0}{dy_0}, \quad \cos Z=\frac{1}{k_0};$$

itemque

$$\cos X = -\frac{1}{k_n}\frac{dz_n}{dx_n},\quad \cos Y = -\frac{1}{k_n}\frac{dz_n}{dx_n},\quad \cos Z = \frac{1}{k_n}\ .$$

Ex quibus colligimus (109) minimam rectarum omnium, quae duci possunt ab una superficie curva ad alteram, fore normalem utrique superficiei in punctis, quorum coordinatae x_0, y_0, z_0, x_n, y_n, z_n determinantur aequationibus (h), (h'), (h'').

II.° Ex lineis jungentibus duo puncta data in spatio determinare eam, in qua

$$\int_{x_0}^{x_n} \sqrt{\left(\frac{1+y'^2+z'^2}{x}\right)}dx$$

est maximum minimumve. Habemus

$$R=0\ ,\ S=\frac{y'}{\sqrt{x}\sqrt{(1+y'^2+z'^2)}},\ T=0,\ V=0,\ldots$$

$$r=0\ ,\ s=\frac{z'}{\sqrt{x}\sqrt{(1+y'^2+z'^2)}},\ t=0,\ v=0,\ldots;$$

et binae (b^{IV}), (b^{XIII}) evadunt

$$\frac{dS}{dx}=0\ ,\ \frac{ds}{dx}=0\ :$$

igitur

$$S=\frac{y'}{\sqrt{x}\sqrt{(1+y'^2+z'^2)}}=C,\quad s=\frac{z'}{\sqrt{x}\sqrt{(1+y'^2+z'^2)}}=C_2.$$

Ex his eruitur

$$\frac{y'}{z'}=C_3,\ \text{seu}\ dy=C_3dz\ ;$$

unde

$$y = C_1 z + C_2,$$

aequatio ad projectionem lineae quaesitae in plano YAZ : ergo linea illa est plana. Constituatur planum XAY in plano ipsius lineae ut sit $z=0$, et consequenter $z'=0$; aequatio ad lineam istam erit

$$\frac{y'}{\sqrt{x}\sqrt{(1+y'^2)}} = C;$$

eadem nimirum, quae jam prodiit (227. IV.°).

DE CALCULO DIFFERENTIARUM FINITARUM; ET DE AEQUATIONIBUS AD DIFFERENTIAS MISTAS.

DE DIFFERENTIIS FINITIS POTENTIARUM INTEGRARUM UNIUS VARIABILIS; NEC NON DE ISTIUSMODI POTENTIARUM INTEGRALIBUS QUOAD DIFFERENTIAS FINITAS.

231. Designantes per $\Delta^m x^n$ differentiam $m$simam potentiae x^n, sic eam determinabimus ope calculi residuorum. Habemus (67. 5.°)

$$\mathcal{E}\frac{e^{xz}}{((z^{n+1}))}=\frac{x^n}{1.2.3...n}, \text{ unde } x^n=1.2.3...n\,\mathcal{E}\frac{e^{xz}}{((z^{n+1}))} \ldots (i);$$

et consequenter

$$\Delta^m x^n = 1.2.3...n\,\mathcal{E}\frac{\Delta^m{}_x e^{xz}}{((z^{n+1}))} \ldots (i').$$

Constans atque arbitrarium incrementum Δx exhibeatur compendii causa per α; erit

$$\left.\begin{array}{l}
\Delta_x e^{xz}=e^{(x+\alpha)z}-e^{xz}=e^{xz}(e^{\alpha z}-1), \\
\text{et simili modo } \Delta^2{}_x e^{xz}=e^{xz}(e^{\alpha z}-1)^2, \\
\Delta^3{}_x e^{xz}=e^{xz}(e^{\alpha z}-1)^3, \ldots \Delta^m{}_x e^{xz}=e^{xz}(e^{\alpha z}-1)^m:
\end{array}\right\} (i'')$$

hinc mutabitur (i') in

$$\Delta^m x^n = 1.2.3 \ldots n\,\mathcal{E}\frac{e^{xz}(e^{\alpha z}-1)^m}{((z^{n+1}))} \ldots (i''').$$

Jamvero (241 ex p. 1.^{a})

$$e^{xz}=1+xz+\frac{x^2z^2}{2}+\frac{x^3z^3}{2.3}+\ldots,\ (e^{\alpha z}-1)^m=\alpha^m z^m\left(1+\frac{\alpha z}{2}+\frac{\alpha^2z^2}{2.3}+\ldots\right)^m,$$

ac proinde

$$\frac{e^{xz}(e^{\alpha z}-1)^m}{z^{n+1}}=\frac{\alpha^m z^{m-n}}{z}\left(1+xz+\frac{x^2z^2}{2}+\frac{x^3z^3}{2.3}+\ldots\right)\left(1+\frac{\alpha z}{2}+\frac{\alpha^2z^2}{2.3}+\ldots\right)^m;$$

igitur (65), si

$$\alpha^m\left(1+xz+\frac{x^2z^2}{2}+\frac{x^3z^3}{2.3}+\ldots\right)\left(1+\frac{\alpha z}{2}+\frac{\alpha^2z^2}{2.3}+\ldots\right)^m\ldots\ (i^{\text{IV}})$$

evolvitur secundum potentias variabilis z, residuum

$$\mathcal{E}\frac{e^{xz}(e^{\alpha z}-1)^m}{((z^{n+1}))}$$

nihil erit aliud nisi coefficiens potentiae z^{n-m}. Pone

$$\left(1+\frac{z}{2}+\frac{z^2}{2.3}+\ldots\right)^m=1+A_1z+A_2z^2+A_3z^3+\ldots\ (i^{\text{V}});$$

innotescent A_1, A_2, A_3, ... ex dictis (35. 2.°): vertitur autem (i^{IV}) in

$$\alpha^m\left(1+xz+\frac{x^2z^2}{2}+\frac{x^3z^3}{2.3}+\ldots\right)(1+A_1\alpha z+A_2\alpha^2z^2+A_3\alpha^3z^3+\ldots);$$

itaque

$$\mathcal{E}\frac{e^{xz}(e^{\alpha z}-1)^m}{((z^{n+1}))}=\alpha^m\Big[\frac{1}{1.2.3\ldots(n-m)}x^{n-m}+$$

$$\frac{1}{1.2.3\ldots(n-m-1)}A_1\alpha x^{n-m-1}+\frac{1}{1.2.3\ldots(n-m-2)}A_2\alpha^2x^{n-m-2}+\ldots+$$

$$\frac{1}{1}A_{n-m-1}\alpha^{n-m-1}x+A_{n-m}\alpha^{n-m}\Big];$$

et consequenter

$$\left.\begin{array}{l}\Delta^m x^n = n(n-1)\ldots(n-m+1)\alpha^m x^{n-m} + \\ n(n-1)\ldots(n-m)A_1\alpha^{m+1}x^{n-m-1} + \\ n(n-1)\ldots(n-m-1)A_2\alpha^{m+2}x^{n-m-2} + \ldots + \\ n(n-1)\ldots 2A_{n-m-1}\alpha^{n-1}x + n(n-1)\ldots 2.1A_{n-m}\alpha^n.\end{array}\right\}(i^{VI})$$

Fac 1.° $n=m$, 2.° $n=m+1$, 3.° $n=m+2$, et caet...;

provenient

$$\Delta^m x^m = m(m-1)\ldots 2.1\,\alpha^m,\ \Delta^m x^{m+1} =$$
$$(m+1)m\ldots 2\alpha^m x + (m+1)m\ldots 2.1A_1\alpha^{m+1},\ \Delta^m x^{m+2} =$$
$$(m+2)(m+1)\ldots 3\alpha^m x^2 + (m+2)(m+1)\ldots 2A_1\alpha^{m+1}x$$
$$+(m+2)(m+1)\ldots 2.1A_2\alpha^{m+2},\ \text{et caet.}\ldots;$$

in quibus substituendi valores A_1, A_2, A_3, ... ex (i^V), videlicet

$$A_1 = \frac{m}{2},\ A_2 = \frac{m(3m+1)}{24},\ A_3 = \frac{m^2(m+1)}{48},\ \ldots$$

232. In aequatione

$$\Delta F(x) = f(x)$$

functio $F(x)$ vocatur integrale functionis $f(x)$ quoad differentias finitas, designaturque per $\Sigma f(x)$. Pone $f(x)=x^n$; habebis

$$\Delta F(x) = x^n,$$

ideoque (231. i)

$$F(x) = \Sigma x^n = 1.2\ldots n\Sigma_x \mathfrak{E}\frac{e^{xz}}{((z^{n+1}))}.$$

Sed

$$\mathcal{E}\frac{e^{xz}}{((z^{n+1}))}=\mathcal{E}\frac{1}{e^{\alpha z}-1}\frac{\Delta_x e^{xz}}{((z^{n+1}))}=\Delta_x\mathcal{E}\frac{1}{e^{\alpha z}-1}\frac{e^{xz}}{((z^{n+1}))},$$

ac proinde

$$\Sigma_x\,\mathcal{E}\frac{e^{xz}}{((z^{n+1}))}=\mathcal{E}\frac{1}{e^{\alpha z}-1}\frac{e^{xz}}{((z^{n+1}))};$$

igitur

$$\Sigma x^n=1.2\ldots n\,\mathcal{E}\frac{1}{e^{\alpha z}-1}\frac{e^{xz}}{((z^{n+1}))}+\mathrm{C}\ldots(i^{\mathrm{VII}}).$$

Est autem

$$\frac{1}{e^{\alpha z}-1}=(e^{\alpha z}-1)^{-1}=\frac{1}{\alpha z}\left(1+\frac{\alpha z}{2}+\frac{\alpha^2 z^2}{2.3}+\ldots\right)^{-1},$$

ideoque

$$\frac{1}{e^{\alpha z}-1}\frac{e^{xz}}{z^{n+1}}=\frac{\alpha^{-1}z^{-n-1}}{z}\left(1+xz+\frac{x^2z^2}{2}+\frac{x^3z^3}{2.3}+\ldots\right)\left(1+\frac{\alpha z}{2}+\frac{\alpha^2 z^2}{2.3}+\ldots\right)^{-1};$$

igitur (65) si

$$\alpha^{-1}\left(1+xz+\frac{x^2z^2}{2}+\frac{x^3z^3}{2.3}+\ldots\right)\left(1+\frac{\alpha z}{2}+\frac{\alpha^2z^2}{2.3}+\ldots\right)^{-1}\ \ldots(i^{\mathrm{VIII}})$$

evolvitur secundum potentias variabilis z, residuum

$$\mathcal{E}\frac{1}{e^{\alpha z}-1}\frac{e^{xz}}{((z^{n+1}))}$$

nihil erit aliud nisi coefficiens potentiae z^{n+1}. Pone

$$\left(1+\frac{z}{2}+\frac{z^2}{2.3}+\ldots\right)^{-1}=1+\mathrm{B}_1z+\mathrm{B}_2z^2+\mathrm{B}_3z^3+\ldots(i^{\mathrm{IX}});$$

vertetur (i^{VIII}) in

$$\alpha^{-1}\left(1+xz+\frac{x^2z^2}{2}+\frac{x^3z^3}{2.3}+\ldots\right)\left(1+B_1\alpha z+B_2\alpha^2z^2+B_3\alpha^3z^3+\ldots\right).$$

Quare

$$\Sigma\frac{1}{e^{\alpha z}-1}\,\frac{e^{xz}}{((z^{n+1}))}=\alpha^{-1}\left(\frac{x^{n+1}}{2.3\ldots(n+1)}+B_1\alpha\frac{x^n}{2.3\ldots n}+\right.$$

$$B_2\alpha^2\frac{x^{n-1}}{2.3\ldots(n-1)}+B_3\alpha^3\frac{x^{n-2}}{2.3\ldots(n-2)}+\ldots$$

$$\left.+B_{n-1}\alpha^{n-1}\frac{x^2}{2}+B_n\alpha^n x+B_{n+1}\alpha^{n+1}\right);$$

et consequenter

$$\Sigma x^n=\frac{x^{n+1}}{(n+1)\alpha}+B_1x^n+nB_2\alpha x^{n-1}+n(n-1)B_3\alpha^2x^{n-2}+\ldots+C\;(i^{\mathrm{x}}):$$

formula (i^{IX}) praebet (35. 2.°)

$$B_1=-\tfrac{1}{2},\ B_2=\frac{1}{2}.\frac{1}{6},\ B_3=0,\ B_4=\frac{1}{1.2.3.4}\cdot\frac{1}{30},\ B_5=0,\ \text{et caet}\ldots$$

Hinc facto $n=0\ ,\ 1\ ,\ 2\ ,\ 3\ ,\ \ldots$ prodibunt

$$\left.\begin{array}{l}\Sigma 1=\frac{x}{\alpha}+C\ ,\ \Sigma x=\frac{x^2}{2\alpha}-\frac{x}{2}+C\ ,\ \Sigma x^2=\frac{x^3}{3\alpha}-\frac{x^2}{2}+\\ \frac{\alpha x}{6}+C\ ,\ \Sigma x^3=\frac{x^4}{4\alpha}-\frac{x^3}{2}+\frac{\alpha x^2}{4}+C\ ,\ \text{et caet.}\ldots\end{array}\right\}(i^{\mathrm{XI}})$$

Ab (i^{VII}) transitur ad

$$\Sigma\Sigma x^n=1.2\ldots n\Sigma\frac{1}{e^{\alpha z}-1}\,\frac{\Sigma_x e^{xz}}{((z^{n+1}))}+C\Sigma 1\,;$$

seu ob $(i^{\mathrm{II}}:i^{\mathrm{XI}})$

$$\Sigma\Sigma x^n = 1.2\ldots n \,\mathcal{E}\, \frac{1}{(e^{\alpha z}-1)^2}\,\frac{e^{xz}}{((z^{n+1}))} + C_1 x + C_2 ;$$

simili modo

$$\Sigma\Sigma\Sigma x^n = 1.2\ldots n \,\mathcal{E}\, \frac{1}{(e^{\alpha z}-1)^3}\,\frac{e^{xz}}{((z^{n+1}))} + C_1 x^2 + C_2 x + C_3 ;$$

et generatim, posito integrationum numero $=m$,

$$\left.\begin{array}{l} \Sigma\Sigma\Sigma\ldots x^n = 1.2\ldots n \,\mathcal{E}\, \dfrac{1}{(e^{\alpha z}-1)^m}\,\dfrac{e^{xz}}{((z^{n+1}))} + \\ C_1 x^{m-1} + C_2 x^{m-2} + \ldots + C_{m-1} x + C_m. \end{array}\right\} (i^{xII})$$

Jamvero

$$\frac{1}{(e^{\alpha z}-1)^m}\,\frac{e^{xz}}{z^{n+1}} = \frac{\alpha^{-m} z^{-n-m}}{z}\left(1 + xz + \frac{x^2 z^2}{2} + \frac{x^3 z^3}{2.3} + \ldots\right)\left(1 + \frac{\alpha z}{2} + \frac{\alpha^2 z^2}{2.3} + \ldots\right)^{-m};$$

facto igitur

$$\left(1 + \frac{z}{2} + \frac{z}{2.3} + \ldots\right)^{-m} = 1 + H_1 z + H_2 z^2 + H_3 z^3 + \ldots (i^{xIII}),$$

et evoluto

$$\alpha^{-m}\left(1 + xz + \frac{x^2 z^2}{2} + \frac{x^3 z^3}{2.3} + \ldots\right)(1 + H_1 \alpha z + H_2 \alpha^2 z^2 + \ldots),$$

residuum

$$\mathcal{E}\, \frac{1}{(e^{\alpha z}-1)^m}\,\frac{e^{xz}}{((z^{n+1}))}$$

nihil erit aliud nisi coefficiens (65) potentiae z^{n+m}, nimirum

$$\alpha^{-m}\left[\frac{x^{n+m}}{2.3\ldots(n+m)}+\mathrm{H}_1\alpha\frac{x^{n+m-1}}{2,3\ldots(n+m-1)}+\ldots\right.$$

$$\left.\mathrm{H}_2\alpha^2\frac{x^{n+m-2}}{2.3\ldots(n+m-2)}+\ldots+\mathrm{H}_{n+m-1}\alpha^{n+m-1}x+\mathrm{H}_{n+m}\alpha^{n+m}\right];$$

iccirco

$$\left.\begin{aligned}&\Sigma\Sigma\Sigma\ldots x^n=\frac{x^{n+m}}{(n+1)\ldots(n+m)\alpha^m}+\mathrm{H}_1\frac{x^{n+m-1}}{(n+1)\ldots(n+m-1)\alpha^{m-1}}+\ldots\\&+\mathrm{H}_m x^n+n\mathrm{H}_{m+1}\alpha x^{n-1}+\ldots+\mathrm{C}_1x^{m-1}+\mathrm{C}_2x^{m-2}+\ldots+\mathrm{C}_{m-1}x+\mathrm{C}_m:\end{aligned}\right\}(i^{\mathrm{XIV}})$$

formula (i^{XIII}) suppeditat (35. 2.°)

$$\mathrm{H}_1=-\frac{m}{2},\ \mathrm{H}_2=\frac{m(3m-1)}{24},\ \mathrm{H}_3=-\frac{m^2(m-1)}{48},\ \text{et caet...}$$

233. Formula (i^{VI}) sic (13. II.°) potest exprimi

$$\left.\begin{aligned}&\Delta^m x^n=\alpha^m\frac{d^m x^n}{dx^m}+\mathrm{A}_1\alpha^{m+1}\frac{d^{m+1}x^n}{dx^{m+1}}+\mathrm{A}_2\alpha^{m+2}\frac{d^{m+2}x^n}{dx^{m+2}}+\ldots;\\&\text{formula vero } (i^{\mathrm{XIV}}) \text{ scribi potest}\\&\text{in hunc modum (148)}\\&\Sigma\Sigma\Sigma\ldots x^n=\frac{\int\int\int\ldots x^n dx^m}{\alpha^m}+\mathrm{H}_1\frac{\int\int\int\ldots x^n dx^{m-1}}{\alpha^{m-1}}+\ldots\\&\qquad+\mathrm{H}_m x^n+\mathrm{H}_{m+1}\alpha\frac{d(x^n)}{dx}+\ldots\end{aligned}\right\}(i^{\mathrm{XV}})$$

Hinc si derivatae

$\frac{d(x^n)}{dx}$, $\frac{d^2x^n}{dx^2}$, $\frac{d^3x^n}{dx^3}$, ... designantur per $\mathrm{D}x^n$, D^2x^n, D^3x^n, ...

item integralia quoad differentias finitas

Σx^n, $\Sigma\Sigma x^n$, $\Sigma\Sigma\Sigma x^n$, ... per $\Delta^{-1}x^n$, $\Delta^{-2}x^n$, $\Delta^{-3}x^n$, ...

et integralia

$$\int x^n dx\,,\ \int\int x^n dx^2\,,\ \int\int\int x^n dx^3\,,\ \ldots$$

per $\qquad D^{-1}x^n\,,\ D^{-2}x^n\,,\ D^{-3}x^n\,,\ \ldots,$

exsistent

$$\left.\begin{array}{l}\Delta^m x^n = \alpha^m D^m x^n + A_1\alpha^{m+1}D^{m+1}x^n + \\ A_2\alpha^{m+2}D^{m+2}x^n + \ldots,\ \Delta^{-m}x^n = \alpha^{-m}D^{-m}x^n + \\ H_1\alpha^{-m+1}D^{-m+1}x^n + \ldots + H_m x^n + H_{m+1}\alpha D x^n + \ldots\end{array}\right\}(i^{XVI})$$

DE DIFFERENTIIS FINITIS FUNCTIONUM UNIUS, VEL PLURIUM VARIABILIUM INDEPENDENTIUM; NECNON DE ISTIUSMODI FUNCTIONUM INTEGRALIBUS QUOAD DIFFERENTIAS FINITAS.

234. Denotantibus k, k', k'', k''', ... coefficientes datos, prima (i^{XVI}) suppeditat

$$\Delta^m kx^n = \alpha^m D^m kx^n + A_1\alpha^{m+1}D^{m+1}kx^n + A_2\alpha^{m+2}D^{m+2}kx^n + \ldots,$$

$$\Delta^m k'x^{n+1} = \alpha^m D^m k'x^{n+1} + A_1\alpha^{m+1}D^{m+1}k'x^{n+1} + A_2\alpha^{m+2}D^{m+2}k'x^{n+1} + \ldots,$$

$$\Delta^m k''x^{n+2} = \alpha^m D^m k''x^{n+2} + A_1\alpha^{m+1}D^{m+1}k''x^{n+2} + A_2\alpha^{m+2}D^{m+2}k''x^{n+2} + \ldots,\ \text{et caet.} \ldots;$$

secunda (a^{XVI}) praebet

$$\Delta^{-m}kx^n = \alpha^{-m}D^{-m}kx^n + H_1\alpha^{-m+1}D^{-m+1}kx^n + \ldots + H_m kx^n + H_{m+1}\alpha Dkx^n + \ldots,$$

$$\Delta^{-m}k'x^{n+1} = \alpha^{-m}D^{-m}k'x^{n+1} + H_1\alpha^{-m+1}D^{-m+1}k'x^{n+1} + \ldots + H_m k'x^n + H_{m+1}\alpha Dk'x^n + \ldots,$$

$$\Delta^{-m}k''x^{n+2} = \alpha^{-m}D^{-m}k''x^{n+2} + H_1\alpha^{-m+1}D^{-m+1}k''x^{n+2} + \ldots + H_m k''x^n + H_{m+1}\alpha Dk''x^n + \ldots,\ \text{et caet.} \ldots$$

Quare

$\Delta^m(kx^n+k'x^{n+1}+k''x^{n+2}+\dots)=\alpha^m D^m(kx^n+k'x^{n+1}+$
$k''x^{n+2}+\dots)+A_1\alpha^{m+1}D^{m+1}(kx^n+k'x^{n+1}+k''x^{n+2}+\dots)$
$+A_2\alpha^{m+2}D^{m+2}(kx^n+k'x^{n+1}+k''x^{n+2}+\dots)+\dots;$
$\Delta^{-m}(kx^n+k'x^{n+1}+k''x^{n+2}+\dots)=\alpha^{-m}D^{-m}(kx^n+k'x^{n+1}+$
$k''x^{n+2}+\dots)+H_1\alpha^{-m+1}D^{-m+1}(kx^n+k'x^{n+1}+k''x^{n+2}+\dots)+\dots$
$+H_m(kx^n+k'x^{n+1}+k''x^{n+2}+\dots)+H_{m+1}\alpha D(kx^n+$
$k'x^{n+1}+k''x^{n+2}+\dots)+\dots;$

seu, facto

$$kx^n+k'x^{n+1}+k''x^{n+2}+\dots=\chi,$$

$$\left.\begin{aligned}&\Delta^m\chi=\alpha^m D^m\chi+A_1\alpha^{m+1}D^{m+1}\chi+A_2\alpha^{m+2}D^{m+2}\chi+\dots,\\&\Delta^{-m}\chi=\alpha^{-m}D^{-m}\chi+H_1\alpha^{-m+1}D^{-m+1}\chi+\dots+H_m\chi+H_{m+1}\alpha D\chi+\dots\end{aligned}\right\}(i_1)$$

235. Aequatio (i^{v} 231) traducitur (241 ex p. 1.ª) ad

$$(e^z-1)^m=z^m(1+A_1z+A_2z^2+\dots)=z^m+A_1z^{m+1}+A_2z^{m+2}+\dots;$$

aequatio vero (i^{XIII} 232) ad

$$(e^z-1)^{-m}=z^{-m}(1+H_1z+H_2z^2+\dots)=z^{-m}+H_1z^{-m+1}+H_2z^{-m+2}+\dots;$$

et adhibito αD pro z,

$$(e^{\alpha D}-1)^m=\alpha^m D^m+A_1\alpha^{m+1}D^{m+1}+A_2\alpha^{m+2}D^{m+2}+\dots,$$

$$(e^{\alpha D}-1)^{-m}=\alpha^{-m}D^{-m}+H_1\alpha^{-m+1}D^{-m+1}+H_2\alpha^{-m+2}D^{-m+2}+\dots$$

Hinc (234. i_1)

$$\Delta^m\chi=(e^{\alpha D}-1)^m\chi,\quad \Delta^{-m}\chi=(e^{\alpha D}-1)^{-m}\chi\;\dots\;(i_2);$$

modo tamen, facta in seriem evolutione secundum potentias quantitatis α, in terminis inde prodeuntibus, et sese exhibentibus sub forma

$$A_r\alpha^{m+r}D^{m+r}\chi,\quad H_r\alpha^{r-m}D^{r-m}\chi,$$

spectetur D ut supra (233) : quod etiam in sequenti n.° animadvertendum.

236. Haec notentur : 1.° quoniam

$$\Delta[bx+h][b(x+\alpha)+h][b(x+2\alpha)+h]\dots[b(x+n\alpha)+h]=$$

$$(n+1)b\alpha[b(x+\alpha)+h][b(x+2\alpha)+h]\dots[b(x+n\alpha)+h],$$

ideo

$$\Sigma[b(x+\alpha)+h][b(x+2\alpha)+h]\dots[b(x+n\alpha)+h]=$$

$$\frac{[bx+h][b(x+\alpha)+h][b(x+2\alpha)+h]\dots[b(x+n\alpha)+h]}{(n+1)b\alpha}+C_1.$$

2.° quia

$$\Delta\frac{1}{[bx+h][b(x+\alpha)+h]\dots[b(x+n\alpha)+h]}=$$

$$-\frac{(n+1)b\alpha}{[bx+h][b(x+\alpha)+h]\dots[b(x+(n+1)\alpha)+h]},$$

iccirco

$$\Sigma\frac{1}{[bx+h][b(x+\alpha)+h]\dots[b(x+(n+1)\alpha)+h]}=$$

$$-\frac{1}{(n+1)b\alpha}\cdot\frac{1}{[bx+h][b(x+\alpha)+h]\dots[b(x+n\alpha)+h]}+C_1.$$

3.° data functione

$$a^x(b+b'x+b''x^2+\dots),$$

pone

$$a^x(b+b'x+b''x^2+\dots)=\Delta a^x(g+g'x+g''x^2+\dots),$$

seu

$$b+b'x+b''x^2+\dots=a^\alpha[g+g'(x+\alpha)+g''(x+\alpha)^2+\dots]-g-g'x-g''x^2-\dots;$$

ad g, g', g'', ... determinandas habebis (131. 8.° ex p. 1a.)

$$a^{\alpha}(g+\alpha g'+\alpha^2 g''+\ldots)-g=b\,,\ a^{\alpha}(g'+2\alpha g''+3\alpha^2 g'''+\ldots)-g'=b',$$

et caet.....; eritque

$$\Sigma a^x(b+b'x+b''x^2+\ldots\ldots)=a^x(g+g'x+g''x^2+\ldots).$$

Sint v. g. $b''=0$, $b'''=0$, et caet..., ac proinde $g''=0$, $g'''=0$, et caet...; provenient

$$a^{\alpha}(g+\alpha g')-g=b\,,\ a^{\alpha}g'-g'=b',$$

unde

$$g'=\frac{b'}{a^{\alpha}-1}\,,\ g=\frac{a^{\alpha}(b-\alpha b')-b}{(a^{\alpha}-1)^2}\,;$$

et consequenter

$$\Sigma a^x(b+b'x)=a^x\Big[\frac{a^{\alpha}(b-\alpha b')-b}{(a^{\alpha}-1)^2}+\frac{b'x}{a^{\alpha}-1}\Big]+C_1.$$

4.° habemus

$$\Delta L[x(x-\alpha)(x-2\alpha)\ldots(x-(n-1)\alpha)]=L\Big[\frac{x+\alpha}{x-(n-1)\alpha}\Big]:$$

propterea

$$\Sigma L\Big[\frac{x+\alpha}{x-(n-1)\alpha}\Big]=L[C_1x(x-\alpha)(x-2\alpha)\ldots(x-(n-1)\alpha)].$$

Hinc quoad valorem x *nplum* incrementi α erit

$$\Sigma L\Big[\frac{x+\alpha}{\alpha}\Big]=L[C_1x(x-\alpha)(x-2\alpha)\ldots 2\alpha.\alpha]\,;$$

et facto $\alpha=1$.

$$\Sigma L(x+1)=L[C_1x(x-1)(x-2)\ldots 2.1].$$

5.° functio $\psi(x)$ praebens $\Delta\psi(x)=0$ vocatur *periodica*; habes exempla in

$$cos\frac{2\pi x}{\alpha}, cos^2(\frac{2\pi x}{\alpha}), \ldots \frac{1}{cos\frac{2\pi x}{\alpha}}, \frac{1}{cos^2(\frac{2\pi x}{\alpha})}, \ldots L(cos\frac{2\pi x}{\alpha}), \ldots :$$

ubi ergo sermo sit de integralibus quoad differentias finitas, pro constantibus arbitrariis C_1, C_2, . . . poterunt adhiberi functiones periodicae itidem arbitrariae. Hoc pacto generalitati magis consuletur.

237. Sit $f(x, y, z, \ldots)$ functio integra variabilium independentium $x, y, z, \ldots$; factis $\Delta x=\alpha$, $\Delta y=\alpha'$, $\Delta z=\alpha''$, . . . , prima (i_2. 235) suppeditabit

$$\Delta^m{}_x f=(e^{\alpha D_x}-1)^m f, \; \Delta^n{}_y f=(e^{\alpha' D_y}-1)^n f, \text{ et caet. } \ldots;$$

secunda (i_2. 235) praebebit

$$\Delta^{-m}{}_x f=(e^{\alpha D_x}-1)^{-m} f, \; \Delta^{-n}{}_y f=(e^{\alpha' D_y}-1)^{-n} f, \text{ et caet.} \ldots:$$

propterea

$$\left.\begin{aligned} \Delta^m{}_x \Delta^n{}_y \ldots f&=(e^{\alpha D_x}-1)^m (e^{\alpha' D_y}-1)^n \ldots f, \\ \Delta^{-m}{}_x \Delta^{-n}{}_y \ldots f&=(e^{\alpha D_x}-1)^{-m}(e^{\alpha' D_y}-1)^{-n} \ldots f. \end{aligned}\right\} (i_3)$$

Ad haec: pone

$$f(x, y, z, \ldots)=u;$$

ob primam (i_2. 235) erit

$$f(x+\alpha, y, z, \ldots)=u+\Delta_x u=u+e^{\alpha D_x}u-u=e^{\alpha D_x}u,$$

ideoque

$$f(x+\alpha, y+\alpha', z, \ldots)=e^{\alpha' D_y} f(x+\alpha, y, z, \ldots)=$$

$e^{\alpha' D_y} e^{\alpha D_x} u = e^{\alpha D_x + \alpha' D_y} u$, $f(x+\alpha, y+\alpha', z+\alpha'', \ldots) =$
$e^{\alpha'' D_z} f(x+\alpha, y+\alpha', z, \ldots) = e^{\alpha D_x + \alpha' D_y + \alpha'' D_z} u$,
et caet. . . . :
quicumque igitur sit numerus variabilium independentium, exsistet

$$f(x+\Delta x, y+\Delta y, z+\Delta z, \ldots) = e^{\alpha D_x + \alpha' D_y + \alpha'' D_z + \ldots} u \ldots (i_k).$$

Hinc

$\Delta u = f(x+\Delta x, y+\Delta y, z+\Delta z, \ldots) - f(x, y, z, \ldots) =$
$(e^{\alpha D_x + \alpha' D_y + \alpha'' D_z + \ldots} - 1)u$, $\Delta^2 u = f(x+2\Delta x, y+2\Delta y,$
$z+2\Delta z, \ldots) - 2f(x+\Delta x, y+\Delta y, z+\Delta z, \ldots) + f(x, y, z, \ldots) =$
$(e^{2(\alpha D_x + \alpha' D_y + \alpha'' D_z + \ldots)} - 2e^{\alpha D_x + \alpha' D_y + \alpha'' D_z + \ldots} + 1)u =$
$(e^{\alpha D_x + \alpha' D_y + \alpha'' D_z + \ldots} - 1)^2 u$, $\Delta^3 u = f(x+3\Delta x, \ldots) -$
$3f(x+2\Delta x, \ldots) + 3f(x+\Delta x, \ldots) - f(x, \ldots) =$
$(e^{3(\alpha D_x + \ldots)} - 3e^{2(\alpha D_x + \ldots)} + 3e^{\alpha D_x + \ldots} - 1)u = (e^{\alpha D_x + \ldots} - 1)^3 u$,
et caet. . . . ; generatim

$$\Delta^m u = (e^{\alpha D_x + \alpha' D_y + \alpha'' D_z + \ldots} - 1)^m u \ldots (i_8).$$

DE QUARUMDAM AEQUATIONUM INTEGRATIONE QUOAD DIFFERENTIAS FINITAS.

238. Proponitur integranda aequatio

$$\Delta y - qy = r \ldots (l)$$

in hypothesi q et r functionum unius variabilis x.
Sumptis novis variabilibus u et v, fac $y = e^u v$;

quem valorem substitue in (l) : prodibit

$$e^{u}v(e^{\Delta u}-1-q)+e^{u+\Delta u}\Delta v=r.$$

Pone

$$e^{\Delta u}-1-q=0\,,\ \text{ideoque}\ e^{u+\Delta u}\Delta v=r\,;$$

istarum prima suppeditat

$$\Delta u=\mathrm{L}(1+q)\,,\quad u=\Sigma\mathrm{L}(1+q)\,,$$

et consequenter secunda praebebit

$$v=\Sigma\frac{r}{e^{\Sigma\mathrm{L}(1+q)+\mathrm{L}(1+q)}}=\Sigma\frac{re^{-\Sigma\mathrm{L}(1+q)}}{e^{\mathrm{L}(1+q)}}=\Sigma\frac{r}{1+q}e^{-\Sigma\mathrm{L}(1+q)}\,;$$

igitur

$$y=e^{\Sigma\mathrm{L}(1+q)}\Sigma\frac{r}{1+q}e^{-\Sigma\mathrm{L}(1+q)}\ \ldots\ (l').$$

In hypothesi q constantis, cum ob primam (i^{x1}. 232) exsistant

$$e^{\Sigma\mathrm{L}(1+q)}=[e^{\mathrm{L}(1+q)}]^{\Sigma 1}=(1+q)^{\frac{x}{\alpha}}\,,$$

$$\frac{1}{1+q}e^{-\Sigma\mathrm{L}(1+q)}=\frac{1}{1+q}[e^{\mathrm{L}(1+q)}]^{-\Sigma 1}=\frac{(1+q)^{-\frac{x}{\alpha}}}{1+q}\,,$$

vertetur (l') in

$$y=(1+q)^{\frac{x}{\alpha}-1}\Sigma r(1+q)^{-\frac{x}{\alpha}}\ \ldots\ (l'').$$

Iuvat duo hic subjicere problemata.

I.° Determinare quoties possint inter se permu-

tari x litterae. Quaesitus permutationum numerus designetur per y : et facto $\alpha=1$, ut litterarum numerus evadat $x+1$, proveniet

$$\Delta y = xy\ ;$$

siquidem ad obtinendas permutationes inter $x+1$ litteras a, b, ... h, k satis est in singulis permutationibus inter x litteras a, b, ... h ponere novam litteram k vel primo, vel secundo, vel tertio et caet., vel postremo loco. Erunt itaque

$$q = x,\ r = 0\ ;$$

et consequenter (236. 4.°)

$$y = Ce^{\Sigma L(1+x)} = Ce^{L[C_1 x(x-1)(x-2)..2.1]} = C_2 x(x-1)(x-2)...2.1$$

Ad C_2 quod spectat, cum in hypothesi unius litterae unica obtineatur permutatio, erit igitur $C_2=1$. Recole p.^{em} 1.^{am} n.° 111. I.°

II.° Ex urna, in qua continebantur a globuli albi et b nigri, pone jam esse extractos $n-1$ albos et $m-x$ nigros : globulis haud repositis in urnam, quaeritur probabilitas extrahendi adhuc n^{simum} album priusquam extrahantur x nigri. Quaesita eventus probabilitas designetur per $F(x)$; ea vertetur in $F(x-1)$, si in prima subsequentium extractionum sese exhibeat globulus niger. Ad haec : eventui favent $a-(n-1)$ casus, adversantur $b-(m-x)$; qui tamen $b-(m-x)$ contrarii casus adducunt probabilitatem $F(x-1)$: igitur (220. 1.° 3.° 4.° ex p. 1.^{a})

$$F(x) = \frac{a-(n-1)}{[a-(n-1)]+[b-(m-x)]} + \frac{b-(m-x)}{[a-(n-1)]+[b-(m-x)]} F(x-1).$$

Substitue $x+1$ loco x, et compendii causa fac

$$a-(n-1)=g\ ,\ b-m=g'\ ,\ g+g'=g''\ ;$$

habebis

$$F(x+1) = \frac{g}{g''+x+1} + \frac{g'+x+1}{g''+x+1}F(x).$$

Sume $\Delta x = \alpha = 1$, $F(x) = y$; erit $F(x+1) = y + \Delta y$. Hinc

$$\Delta y + \frac{g}{g''+x+1}y = \frac{g}{g''+x+1};$$

cujus integrale obtinetur ex (l'), positis

$$q = -\frac{g}{g''+x+1}, \quad r = \frac{g}{g''+x+1}.$$

239. Si aequatio n^{simi} ordinis quoad differentias finitas

$$\Delta^n y + a_1 \Delta^{n-1} y + a_2 \Delta^{n-2} y + \dots + a_{n-1} \Delta y + a_n y = f(x) \dots (l''')$$

exprimitur in hunc modum

$$(\Delta^n + a_1 \Delta^{n-1} + a_2 \Delta^{n-2} + \dots + a_{n-1} \Delta + a_n) y = f(x),$$

designatis per q_1, q_2, ... q_n radicibus aequationis

$$\Delta^n + a_1 \Delta^{n-1} + a_2 \Delta^{n-2} + \dots + a_{n-1} \Delta + a_n = 0,$$

poterit (l''') sic etiam exhiberi

$$(\Delta - q_1)(\Delta - q_2) \dots (\Delta - q_n) y = f(x):$$

unde inferimus integrationem aequationis (l''') traduci ad integrationem n aequationum primi ordinis

$$(\Delta - q_1) y_{n-1} = f(x), \quad (\Delta - q_2) y_{n-2} = y_{n-1},$$

$$(\Delta - q_3) y_{n-3} = y_{n-2}, \dots (\Delta - q_{n-1}) y_1 = y_2, (\Delta - q_n) y = y_1.$$

Sunt autem (238. l'')

$$y_{n-1} = (1+q_1)^{\frac{x}{\alpha}-1} \Sigma (1+q_1)^{-\frac{x}{\alpha}} f(x), \quad y_{n-2} =$$

$$(1+q_2)^{\frac{x}{\alpha}-1}\Sigma(1+q_2)^{-\frac{x}{\alpha}}y_{n-1}\,,\;y_{n-1}=(1+q_2)^{\frac{x}{\alpha}-1}\Sigma(1+q_3)^{-\frac{x}{\alpha}}y_{n-2}\ldots$$

$$y_1=(1+q_{n-1})^{\frac{x}{\alpha}-1}\Sigma(1+q_{n-1})^{-\frac{x}{\alpha}}y_2\,,\;y=(1+q_n)^{\frac{x}{\alpha}-1}\Sigma(1+q_n)^{-\frac{x}{\alpha}}y_1\,;$$

igitur

$$(l^{IV})\quad y=$$

$$\frac{(1+q_n)^{\frac{x}{\alpha}}}{(1+q_n)(1+q_{n-1})\ldots(1+q_1)}\Sigma\left(\frac{1+q_{n-1}}{1+q_n}\right)^{\frac{x}{\alpha}}\Sigma\left(\frac{1+q_{n-2}}{1+q_{n-1}}\right)^{\frac{x}{\alpha}}\ldots\Sigma\left(\frac{1+q_1}{1+q_2}\right)^{\frac{x}{\alpha}}\Sigma\frac{f(x)}{(1+q_1)^{\frac{x}{\alpha}}}:$$

integrale multiplum secundi membri resolvetur in integralia simplicia ope formulae

$$\Sigma s\Delta t=st-\Sigma(t+\Delta t)\Delta s\;\ldots\;(l^{V})\;;$$

quae immediate habetur ex $s\Delta t=\Delta(st)-(t+\Delta t)\Delta s$. Detur v. gr.

$$\Delta^2 y-2\Delta y+y=f(x):$$

erunt $n=2$, $a_1=-2$, $\alpha_2=1$; et aequatio

$$\Delta^2-2\Delta+1=0$$

praebebit $q_1=1$, $q_2=1$. Hinc

$$y=\frac{2^{\frac{x}{\alpha}}}{2.2}\Sigma\left(\frac{2}{2}\right)^{\frac{x}{\alpha}}\Sigma\frac{f(x)}{2^{\frac{x}{\alpha}}}=2^{\frac{x}{\alpha}-2}\Sigma\,[\,\Sigma\,2^{-\frac{x}{\alpha}}f(x)\,].$$

Ad resolvendum integrale duplum in integralia simplicia, sume

$$s=\Sigma 2^{-\frac{x}{\alpha}} f(x), \quad \Delta t=1;$$

ob primam ($i^{\text{xı}}$. 232) exsurget

$$\Sigma[\Sigma 2^{-\frac{x}{\alpha}} f(x)]=\frac{x}{\alpha}\Sigma 2^{-\frac{x}{\alpha}} f(x)-\Sigma\frac{x+\alpha}{\alpha} 2^{-\frac{x}{\alpha}} f(x):$$

propterea

$$y=2^{\frac{x}{\alpha}-2}\left[\frac{x}{\alpha}\Sigma 2^{-\frac{x}{\alpha}} f(x)-\Sigma\frac{x+\alpha}{\alpha} 2^{-\frac{x}{\alpha}} f(x)\right].$$

240. Duo hic obiter notamus: 1.° si aequatio ($g^{\text{vı}}$.187) scribitur (233) ita

$$(D^n+a_1D^{n-1}+a_2D^{n-2}+\ldots+a_{n-1}D+a_n)y=f(x),$$

et radices aequationis

$$D^n+a_1D^{n-1}+a_2D^{n-2}+\ldots+a_{n-1}D+a_n=0$$

dicuntur $q_1, q_2, \ldots q_n$, integratio ipsius ($g^{\text{vı}}$. 187) traducetur ad integrationem (156. I.°) n aequationum primi ordinis

$$(D-q_1)y_{n-1}=f(x), \quad (D-q_2)y_{n-2}=y_{n-1}, \ldots$$
$$(D-q_{n-1})y_1=y_2, \quad (D-q_n)y=y_1.$$

2.° quoniam series recurrens ordinis $n$simi habet terminum generalem t_x sic expressum (229 ex p. 1.a)

$$t_x=A_1t_{x-1}+A_2t_{x-2}+\ldots+A_nt_{x-n},$$

seu, posito $t_x=F(x)$,

$$F(x)=A_1F(x-1)+A_2F(x-2)+\ldots+A_nF(x-n);$$

ideo quoad ejusmodi seriem exsistet

$$F(x+n)=A_1F(x+n-1)+A_2F(x+n-2)+\ldots+A_nF(x).$$

Accepto autem $\Delta x=\alpha=1$, prodeunt

$$F(x+1)=F(x)+\Delta F(x),\ F(x+2)=F(x+1)+\Delta F(x+1)=$$
$$F(x)+2\Delta F(x)+\Delta^2(x),\ \text{et caet.}\ \ldots;$$

itaque investigatio termini generalis t_x traduci poterit ad integrationem aequationis quoad differentias finitas sese exhibentis sub hac forma

$$\Delta^n t_x+a_1\Delta^{n-1}t_x+a_2\Delta^{n-2}t_x+\ldots+a_{n-1}\Delta t_x+a_n t_x=0.$$

241. Proponatur nunc aequatio ad finitas, partialesque differentias

$$\Delta_x z-b\Delta_y z=f(x,y)\ldots(l^{\text{VI}}).$$

Sume $z=\psi(x,y)$: habebis

$\psi(x+\alpha,y)=\psi(x,y)+\Delta_x\psi(x,y)=(1+\Delta_x)\psi(x,y)$,
$\psi(x+2\alpha,y)=\psi(x+\alpha,y)+\Delta_x\psi(x+\alpha,y)=$
$(1+\Delta_x)^2\psi(x,y),\ldots$ et generatim

$$\psi(x+n\alpha,y)=(1+\Delta_x)^n\psi(x,y)\ldots(l^{\text{VII}}):$$

quia insuper (102 ex p. 1.^{a})

$$1-\left(\frac{1+b\Delta_y}{1+\Delta_x}\right)^n=\left[1+\frac{1+b\Delta_y}{1+\Delta_x}+\left(\frac{1+b\Delta_y}{1+\Delta_x}\right)^2+\ldots+\left(\frac{1+b\Delta_y}{1+\Delta_x}\right)^{n-1}\right]\left[1-\frac{1+b\Delta_y}{1+\Delta_x}\right],$$

seu

$$(1+\Delta_x)^n-(1+b\Delta_y)^n=[(1+\Delta_x)^{n-1}+(1+\Delta_x)^{n-2}(1+b\Delta_y)$$
$$+(1+\Delta_x)^{n-3}(1+b\Delta_y)^2+\ldots+(1+b\Delta_y)^{n-1}][\Delta_x-b\Delta_y];$$

iccirco, ob (l^{VI}) et (l^{VII});

$$\psi(x+n\alpha,y)=(1+b\Delta_y)^n\psi(x,y)+[(1+\Delta_x)^{n-1}+$$
$$(1+\Delta_x)^{n-2}(1+b\Delta_y)+\ldots+(1+b\Delta_y)^{n-1}]f(x,y).$$

Aequatio ista manifeste convertitur in

$$\psi(x+n\alpha\,,y)=(1+b\Delta_y)^n\psi(x\,,y)+f[x+(n-1)\alpha\,,y]+ (1+b\Delta_y)f[x+(n-2)\alpha\,,y]+(1+b\Delta_y)^2f[x+(n-3)\alpha,y]+\ldots +(1+b\Delta_y)^{n-1}f(x\,,y)\,,$$

unde

$$\psi(x_0+n\alpha\,,y)=(1+\Delta_y)^n\varphi(y)+f[x_0+(n-1)\alpha\,,y]+ (1+b\Delta_y)f[x_0+(n-2)\alpha\,,y]+(1+b\Delta_y)^2f[x_0+(n-3)\alpha,y]+\ldots +(1+b\Delta_y)^{n-1}f(x_0\,,y)\ldots(l^{\mathrm{VIII}})\,;$$

denotat $\varphi(y)$ valorem illum functionis $\psi(x\,,y)$, qui respondet peculiari valori x_0 quantitatis variabilis x, nimirum

$$\varphi(y)=\psi(x_0\,,y).$$

Ubi ergo spectetur $\varphi(y)$ ut data, poterit per (l^{VIII}) determinari valor incognitae $z=\psi(x\,,y)$ respondens valori $x=x_0+n\alpha$.

242. Si polynomium

$$\Delta^n{}_x+a_1\Delta^{n-1}{}_x\Delta_y+a_2\Delta^{n-2}{}_x\Delta^2{}_y+\ldots+a_{n-1}\Delta_x\Delta^{n-1}{}_y+a_n\Delta^n{}_y$$

resolvitur in factores

$$\Delta_x-b_1\Delta_y\,,\ \Delta_x-b_2\Delta_y\,,\ \ldots\ \Delta_x-b_n\Delta_y\,,$$

ut aequatio ad finitas, partialesque differentias ordinis n^{simi}

$$\left.\begin{array}{l}\Delta^n{}_xz+a_1\Delta^{n-1}{}_x\Delta_yz+a_2\Delta^{n-2}{}_x\Delta^2{}_yz+\ldots\\ \quad+a_{n-1}\Delta_x\Delta^{n-1}{}_yz+a_n\Delta^n{}_yz=f(x\,,y)\end{array}\right\}(l^{\mathrm{IX}})$$

scribi possit in hunc modum

$$(\Delta_x-b_1\Delta_y)(\Delta_x-b_2\Delta_y)\ldots(\Delta_x-b_n\Delta_y)z=f(x\,,y)\,,$$

integratio ipsius (l^{IX}) manifeste traducetur ad integrationem (241) n aequationum

$$(\Delta_x - b_1\Delta_y)z_{n-1} = f(x, y), \ (\Delta_x - b_2\Delta_y)z_{n-2} = z_{n-1}, \dots$$
$$(\Delta_x - b_{n-1}\Delta_y)z_1 = z_2, \ (\Delta_x - b_n\Delta_y)z = z_1.$$

Sic v. gr. integratio aequationis

$$\Delta^2{}_x z - \Delta^2{}_y z = f(x, y)$$

traducitur ad intagrationem binarum

$$(\Delta_x - \Delta_y)z_1 = f(x, y), \ (\Delta_x + \Delta_y)z = z_1.$$

243. Hic quoque (240) obiter notamus illud: si differentialis, partialisque aequatio

$$\left.\begin{aligned} &\frac{d^n z}{dx^n} + a_1\frac{d^n z}{dx^{n-1}dy} + a_2\frac{d^n z}{dx^{n-2}dy^2} + \dots \\ &\quad + a_{n-1}\frac{d^n z}{dx dy^{n-1}} + a_n\frac{d^n z}{dy^n} = f(x, y) \end{aligned}\right\} (l^{\text{x}})$$

scribitur (233) ita

$$(D^n{}_x + a_1 D^{n-1}{}_x D_y + a_2 D^{n-2}{}_x D^2{}_y + \dots$$
$$+ a_{n-1} D_x D^{n-1}{}_y + a_n D^n{}_y)z = f(x, y),$$

et polynomium

$$D^n{}_x + a_1 D^{n-1}{}_x D_y + a_2 D^{n-2}{}_x D^2{}_y + \dots + a_{n-1} D_x D^{n-1}{}_y + a_n D^n{}_y$$

resolvitur in factores

$$D_x - b_1 D_y, \ D_x - b_2 D_y, \ \dots \ D_x - b_n D_y,$$

integratio aequationis (l^{x}) traducetur ad integrationem n aequationum

$$(D_x - b_1 D_y)z_{n-1} = f(x, y), \ (D_x - b_2 D_y)z_{n-2} = z_{n-1}, \dots$$
$$(D_x - b_n D_y)z = z_1.$$

Sic data v. gr.

$$\frac{d^2 z}{dx^2} - 2\frac{d^2 z}{dx dy} + \frac{d^2 z}{dy^2} = x + y,$$

prodibunt

$$(D_x - D_y)z_1 = x + y, \ (D_x - D_y)z = z_1,$$

seu

$$\frac{dz_1}{dx} - \frac{dz_1}{dy} = x + y, \ \frac{dz}{dx} - \frac{dz}{dy} = z_1;$$

ex quibus eruitur (200)

$$z = \frac{x^2}{2}(x+y) + x\varphi(x+y) + \varphi_1(x+y).$$

Recole jam dicta sub n.° 203.

DE INTEGRATIONE QUARUMDAM AEQUATIONUM AD DIFFERENTIAS MISTAS.

244. Sic appellantur aequationes illae, in quibus differentialia sese offerunt utcumque admista cum differentiis finitis: tales sunt v. gr.

$$\Delta y = x\frac{d\Delta y}{dx} - a\left(\frac{d\Delta y}{dx}\right)^2, \ \frac{dy}{dx} = x\Delta\frac{dy}{dx} - a\left(\Delta\frac{dy}{dx}\right)^2. \Big\} \ (o)$$

Ad primam (o) quod spectat, pone $\Delta y = u$; ea mutabitur in

$$u = x\frac{du}{dx} - a\left(\frac{du}{dx}\right)^2 \ldots (o'),$$

ex cujus differentiatione prodit

$$\frac{du}{dx} = \frac{du}{dx} + x\frac{d^2u}{dx^2} - 2a\frac{du}{dx}\frac{d^2u}{dx^2}, \text{ seu } 0 = \left(x - 2a\frac{du}{dx}\right)\frac{d^2u}{dx^2}.$$

Fac 1.° $\frac{d^2u}{dx^2} = 0$; habebis $u = C_1x + C_2$: substituto valore u in (o'), ac dein posita $x = 0$, emerget

$C_2 = -aC^2{}_1$; iccirco

$$\Delta y = C_1 x - aC^2{}_1,$$

unde (232. i^{xi})

$$y = C_1\left(\frac{x^2}{2\alpha} - \frac{x}{2}\right) - aC^2{}_1\frac{x}{\alpha} + C_2.$$

Fac 2.° $x - 2a\frac{du}{dx} = 0$; erit $u = \frac{x^2}{4a} + C_3$: substituto valore u in (o′), ac dein assumpta $x = 0$, prodibit $C_3 = 0$; ideoque $\Delta y = \frac{x^2}{4a}$, et consequenter (232. i^{xi})

$$y = \frac{1}{4a}\left(\frac{x^3}{3\alpha} - \frac{x^2}{2} + \frac{\alpha x}{6}\right) + C;$$

quae nihil exprimet aliud nisi particularem solutionem primae (o).

Ad secundam (o) quod pertinet, pone $\frac{dy}{dx} = v$; ea vertetur in

$$v = x\Delta v - a(\Delta v)^2 \ldots (o'');$$

sumptisque differentiis in hypothesi $\Delta x = \alpha = 1$,

$$o = [x + 1 - a(2\Delta v + \Delta^2 v)]\Delta^2 v.$$

Factor $\Delta^2 v$ praebet (232. i^{xi})

$$\Delta v = C_1,\ v = C\Sigma 1 = C_1 x + C_2:$$

adhibitis substitutionibus in (o″), ac dein facta $x = 0$, emerget $C_2 = -aC^2{}_1$; propterea

$$dy = C_1 x dx - aC^2{}_1 dx,\ \text{et}\ y = \frac{C_1 x^2}{2} - aC^2{}_1 x + C_2.$$

Factor $x+1-a(2\Delta\nu+\Delta^2\nu)$ suppeditat

$$\Delta^2\nu+2\Delta\nu=\frac{x+1}{a},$$

cujus integrale quoad differentias finitas obtinetur ex dictis (239)

245. Ad integrationem confert interdum transformatio in aequationes mere differentiales ordinis infiniti ope formularum (32)

$$\Delta y=f(x+\alpha)-f(x)=\alpha\frac{dy}{dx}+\frac{\alpha^2}{2}\frac{d^2y}{dx^2}+\frac{\alpha^3}{2.3}\frac{d^3y}{dx^3}+\ldots,$$

$$\Delta\frac{dy}{dx}=\Delta f'(x)=f'(x+\alpha)-f'(x)=\alpha\frac{d^2y}{dx^2}+\frac{\alpha^2}{2}\frac{d^3y}{dx^3}+\ldots,$$

et caet. . . . ; sic aequatio

$$\frac{dy}{dx}=a\frac{\Delta y}{\alpha}\ldots(o''')$$

transformatur in

$$(a-1)\frac{dy}{dx}+a\left(\frac{\alpha}{2}\frac{d^2y}{dx^2}+\frac{\alpha^2}{2.3}\frac{d^3y}{dx^3}+\frac{\alpha^3}{2.3.4}\frac{d^4y}{dx^4}+\ldots\right)=0\ldots(o^{IV}).$$

Jam si assumitur $y=Ce^{rx}$, traducetur (o^{IV}) ad

$$(a-1)r+a\left(\frac{\alpha r^2}{2}+\frac{\alpha^2r^3}{2.3}+\frac{\alpha^3r^4}{2.3.4}+\ldots\right)=0\ldots(o^{V}),$$

cui satisfacit $r=0$; tum, divisa (o^{V}) per r,

$$a-1+a\left(\frac{\alpha r}{2}+\frac{\alpha^2r^2}{2.3}+\frac{\alpha^3r^3}{2.3.4}+\ldots\right)=0\ldots(o^{VI});$$

et denotantibus r_1, r_2, r_3, . . . radices aequationis (o^{VI}), erit

$$y=C+C_1e^{r_1x}+C_2e^{r_2x}+C_3e^{r_3x}+\ldots$$

Caeterum ex dictis (72) facile intelligimus unam saltem e radicibus illis v. gr. r_1 fore realem.

246. Ad aliquam theoriae applicationem ostendendam finge tibi curvam planam, et abscissarum x, x_1 differentiam x_1-x pone $=\alpha$; per curvae punctum (x, y) duc tangentem, et per puncta (x, y), (x_1, y_1) duc secantem: fac autem ut distantia k computata in axe abscissarum inter secantem istam et ordinatam y servet constanter eamdem rationem $\frac{a}{1}$ cum subtangente: invenire aequationem ad ejusmodi curvam. Habemus

$$\frac{k}{y}=\frac{x_1-x}{y_1-y}=\frac{\alpha}{\Delta y},\ k=y\frac{\alpha}{\Delta y}\ ;$$

proinde (73)

$$ay\frac{dx}{dy}=y\frac{\alpha}{\Delta y}\ :$$

quae cum recidat in (o'''. 245), dabit

$$y=C+C_1e^{r_1x}\ .$$

247. Sit nunc aequatio ad mistas, partialesque differentias

$$\Delta_x z+a\frac{d_y z}{dy}+bz=0\ldots(o^{VII}).$$

Posita

$$z=C^x e^{c_1 y}\ldots(o^{VIII}),$$

vertetur (o^{VII}) in $c^\alpha-1+ac_1+b=0$, unde $c_1=\frac{1-b-c^\alpha}{a}$; et consequenter

$$z=c^x e^{\frac{1-b-c^\alpha}{a}y}\ldots(o^{IX}).$$

Detur quoque

$$\Delta_x z + a\frac{d_y z}{dy} + b\frac{d_y \Delta_x z}{dy} + gz = 0 \ldots (o^{X}).$$

Adhibitis substitutionibus in (o^{X}) ex (o^{VIII}), assequemur

$$c^{\alpha} - 1 + ac_1 + bc_1(c^{\alpha} - 1) + g = 0,\quad c_1 = \frac{1 - c^{\alpha} - g}{a + b(c^{\alpha} - 1)},$$

ideoque

$$z = c^x e^{\frac{1 - c^{\alpha} - g}{a + b(c^{\alpha} - 1)} y} \ldots (o^{XI}).$$

Formulae (o^{IX}), (o^{XI}) etsi satisfaciunt respondentibus (o^{VII}), (o^{X}), haud tamen debent considerari tamquam integralia completa; non enim continent tot sive constantes arbitrarias, sive functiones arbitrarias, quot sunt ad id necessariae. Possunt equidem ex ipsis (o^{IX}), (o^{XI}) erui integralia illa; sed haec hactenus.

FINIS TERTIAE, AC ULTIMAE PARTIS.

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PRINCIPIA CALCULI DIFFERENTIALIS.

respectivas derivatas : quaeritur utrum in viciniis peculiaris valoris x_n una cum x crescat vel decrescat functio $f(x)$ 17.

Si binae functiones $f(x)$, $\varphi(x)$ et quae ab ipso deducuntur $f'(x)$, $\varphi'(x)$ sunt continuae inter limites x_0, x_n, ac praeterea $\varphi(x)$ vel constanter crescit, vel constanter decrescit ab x_0 ad x_n, erit semper aliquis numerus $\varepsilon < 1$ et > 0 satisfaciens aequationi

$$\frac{f(x_n)-f(x_0)}{\varphi(x_n)-\varphi(x_0)}=\frac{f'(x_0+\varepsilon(x_n-x_0))}{\varphi'(x_0+\varepsilon(x_n-x_0))} \qquad 18.$$

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Ratio determinandi valores functionum unius varialis sese exhibentium sub quibusdam formis indeterminatis : ac 1.° sub forma $\frac{0}{0}$ 26;

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3.° sub formis 0^0, ∞^0, 1^∞ 28.

Si $f(\beta)$ est quantitas infinitesima, cujus basis $= \beta$, ordo $= c$, quaeritur prima inter

$$f(\beta),\ f'(\beta),\ f''(\beta),\ f'''(\beta),\ \ldots$$

haud evanescens una cum β 29.

De maximis, minimisque valoribus functionis continuae $f(x)$: valores x_n, quibus respondet maxima vel minima $f(x_n)$, quaerendi sunt inter radices aequationum

$$f'(x)=0\,,\ \frac{1}{f'(x)}=0:$$

ratio dignoscendi utrum valori x_n reipsa maxima vel minima $f(x_n)$ respondeat 30:

alia ratio 31.

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114	16	$cos B = \frac{sin\varphi}{cos\varphi} = sin B cosc$	$cos B \frac{sin\varphi}{cos\varphi} = sin B cosc.$
115	9	$sin\varphi(C-\varphi)$	$sin(C-\varphi)$
147	24	$a(-cos\,z)$	$a(1-cos\,z)$
153	11	$(n'x)$	$(n''x)$
...	12	$(n'x)$	(nx')
...	25	(*fig.* 82)	(*fig.* 83)
157	28	(66. 182. I.°)	(66 77. 3.°)
158	15	(*fig.* 82.^{a})	(*fig.* 83.^{a})
159	16	(*fig.* 80.^{a})	(*fig.* 81.^{a})
162	9, 22	(*fig.* 80.^{a})	(*fig.* 81.^{a})
166	22	(i)	(i_1)
168	20	192	190
171	20	2CD	2CDE
204	4	C_o	Cz_o
...	14	k	K
...	21	(180. I.° : 1.°)	(180. I.°)
205	6	attenditur	ostenditur

N. B. *pag.* 182, ... 190 pro figuris 83^a, 84^a, 85^a citandae sunt 77^a, 78^a, 79.^{a}

IMPRIMATUR

Fr. Dom. Buttaoni O. P. S. P. A. Mag.

IMPRIMATUR

A. Piatti Archiep. Trapesunt. Vicesg.

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